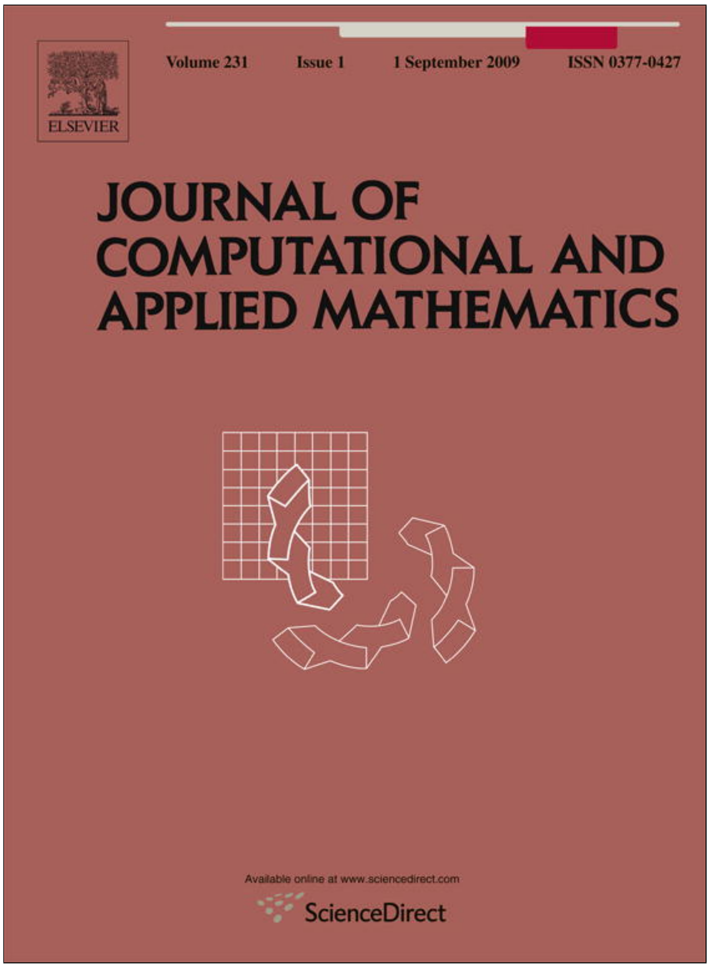
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Quadrature formulas for Fourier coefficients

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ABSTRACT

We consider quadrature formulas of high degree of precision for the computation of the Fourier coefficients in expansions of functions with respect to a system of orthogonal polynomials. In particular, we show the uniqueness of a multiple node formula for the Fourier–Tchebycheff coefficients given by Micchelli and Sharma and construct new Gaussian formulas for the Fourier coefficients of a function, based on the values of the function and its derivatives.

1. Introduction

The approximation of f by partial sums $S_n(f)$ of its series expansion

$$f(t)=\sum_{k=0}^{\infty} a_k(f)P_k(t)$$

with respect to a given system of orthonormal polynomials $\{P_k\}_{k=0}^{\infty}$ is a classical way of recovery of functions. The numerical calculation of the coefficients $a_k(f)$, present in $S_n(f)$, is a main task in such a procedure (see [21]). Recall that if $\{P_k\}_{k=0}^{\infty}$ is a system of orthonormal polynomials on $[a,b]$ with a weight function μ (integrable, non-negative function on $[a,b]$ that vanishes only at isolated points), then

$$a_k(f)=\int_a^b \mu(t)P_k(t)f(t)\mathrm{d}t, \tag{1.1}$$

and the computation of $a_k(f)$ requires the use of a quadrature formula. An application of the Gauss quadrature formula based on n values of the integrand P_kf (with $k<2n-1$) will give the exact result for all polynomials f of degree $2n-1-k$. Is it possible to construct a formula based on n evaluations of f or its derivatives which gives the exact value of the coefficients $a_k(f)$ for polynomials f of higher degree? What is the highest degree of precision that can be attained by a formula based on

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1 Sadly, Dr. Bojanov suddenly passed away during the preparation of this manuscript. As in all previous joint work, he was the major force behind this paper. Dr. Bojanov is no longer among us, but his lifework remains, as does the mark he has made on the mathematical community and the people who knew him. We commemorate him with respect, and with gratitude for all that he has given us. May he rest in peace.

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n evaluations? Studying this question for the coefficients $a_k(f)$ of f with respect to the system of Tchebycheff polynomials of first kind $\{T_k\}_{k=0}^{\infty}$, orthogonal on $[-1, 1]$ with weight $\mu(t) = \frac{1}{\sqrt{1-t^2}}$,

$$T_k(t) = \cos(k \arccos t) = \frac{1}{2^{k-1}}(t-\xi_1)\cdots(t-\xi_k), \quad t \in (-1, 1),$$

Micchelli and Rivlin discovered in [1] the remarkable fact that the quadrature

$$\int_{-1}^{1} \frac{1}{\sqrt{1-t^2}} T_n(t) f(t) \mathrm{d}t \approx \frac{\pi}{n2^n} f'[\xi_1, \ldots, \xi_n] \tag{1.2}$$

is exact for all algebraic polynomials of degree $\leq 3n-1$. Here, $g[x_1, \ldots, x_m]$ denotes the divided difference of g at the points $x_1, \ldots, x_m$, and thus formula (1.2) uses n function values of the derivative f', that is $f'(\xi_1), \ldots, f'(\xi_n)$. It is clear that there is no formula of the form

$$\int_{-1}^{1} \frac{1}{\sqrt{1-t^2}} T_n(t) f(t) \mathrm{d}t \approx \sum_{k=1}^{n} a_k f(x_k) + \sum_{k=1}^{n} b_k f'(x_k) \tag{1.3}$$

which is exact for all polynomials of degree $3n$. The polynomial

$$f(t) = T_n(t)(t-x_1)^2 \cdots (t-x_n)^2$$

is a standard counterexample. Thus the Micchelli–Rivlin formula has the highest degree of precision among all formulas of the type (1.3). Is this extremal formula unique? The question of uniqueness is reduced to the following problem which is also of independent interest: Prove that if Q is a polynomial of degree n with n zeros in $[-1, 1]$ and such that $|Q(\eta_j)| = 1$ at the extremal points $\eta_j = \cos \frac{j\pi}{n}, j = 0, \ldots, n$, of the Tchebycheff polynomial T_n, then $Q \equiv \pm T_n$. This property was proved in [2] and thus the uniqueness of Micchelli–Rivlin quadrature was settled (see [3]).

In this paper, we consider formulas of the type

$$\int_{a}^{b} \mu(t) P_k(t) f(t) \mathrm{d}t \approx \sum_{j=1}^{n} \sum_{i=0}^{\nu_j - 1} c_{ji} f^{(i)}(x_j), \quad a < x_1 < \cdots < x_n < b, \tag{1.4}$$

where ν_j are given natural numbers (*multiplicities*) and $P_k = x^k + \cdots$ is a polynomial of degree k. We say that a number ℓ is the *algebraic degree of precision* (ADP) of (1.4) if (1.4) is exact for all polynomials of degree ℓ and there is a polynomial of degree $\ell + 1$ for which this formula is not exact. Let us denote by $e(\nu)$ the smallest non-negative even integer $\geq \nu$ (clearly $e(\nu) = 0$ for all $\nu \leq 0$), and by $\sigma(P_k)$ the number of zeros of P_k in (a, b) with odd multiplicities. It is easy to see that the ADP (1.4) does not exceed

$$e(\nu_1 - \tau_1) + \cdots + e(\nu_n - \tau_n) + \sigma(P_k) - 1,$$

since the formula is not exact for the polynomial

$$(t - x_1)^{e(\nu_1 - \tau_1)} \ldots (t - x_n)^{e(\nu_n - \tau_n)} (t - t_1) \ldots (t - t_m),$$

where $m = \sigma(P_k)$, $t_1, \ldots, t_m \in (a, b)$, are the zeros of P_k with odd multiplicities, $\tau_i := 1$ if $x_i \in \{t_1, \ldots, t_m\}$ and $\tau_i := 0$ otherwise. Notice that in our applications the polynomial P_k in formula (1.1) for $a_k(f)$ is the kth orthogonal polynomial on $[a, b]$ with weight μ, thus all its zeros are with multiplicity one and we have that $\sigma(P_k) = k$.

Here, for the sake of convenience, we define the formula (1.4) to be *Gaussian*, if it has maximal ADP, that is, if

$$\text{ADP (1.4)} = e(\nu_1 - \tau_1) + \cdots + e(\nu_n - \tau_n) + \sigma(P_k) - 1.$$

For example, formula (1.2) is Gaussian since ADP $(1.2) = 3n - 1$. In this case, $\nu_1 = \cdots = \nu_n = 2$, $\tau_1 = \cdots = \tau_n = 1$, $\sigma(T_n) = n$, and therefore $e(\nu_1 - \tau_1) + \cdots + e(\nu_n - \tau_n) + \sigma(T_n) - 1 = 2n + n - 1 = 3n - 1$. Note, that according to the above definition any interpolatory type quadrature of the form

$$\int_{a}^{b} \mu(t) f(t) \mathrm{d}t \approx \sum_{k=1}^{n} a_k f(x_k) + \sum_{k=1}^{n} b_k f'(x_k)$$

would be also "Gaussian", which is somehow violating our classical understanding about the Gaussian property.

In this paper, we give a complete characterization of the Gaussian formulas of form (1.4) and construct explicitly such formulas in several particular cases. The natural setting is when P_k is the corresponding orthogonal polynomial and one looks for nodes and coefficients that define a Gaussian formula. Another setting is when the nodes $\{x_j\}_{j=1}^{n}$ and their multiplicities $\{\nu_j\}_{j=1}^{n}$ are preassigned, and we seek a polynomial P_k for which the corresponding formula is Gaussian. If Gaussian formulas do not exist, we investigate the formulas of type (1.4) with maximal possible ADP. We also show that there is a one-to-one correspondence between formulas of type (1.4) with simple nodes ($\nu_j = 1, j = 1, \ldots, n$) and fixed orthogonal polynomial P_k and the Gauss–Kronrod formulas [4]. As an application of our main observation (Theorem 2.1) we prove the uniqueness of a formula derived in [5]. This study is a continuation of the approach outlined in [6].

2. General observations

Let us denote by

$$\pi_n(\mathbb{R}) := \left\{ P(t) : P(t) = \sum_{k=0}^{n} d_k t^k, d_k \in \mathbb{R} \right\}$$

the space of all polynomials in one variable of degree at most n. In this section, we discuss general remarks concerning Gaussian quadrature formulas with multiple nodes since, as we shall see below, the study of formulas of type (1.4) for the Fourier coefficients can be reduced to the study of standard multiple node quadratures. The extension of the classical Gauss quadrature to the multiple nodes case took a long time and the effort of many mathematicians. First, Tschakaloff [7] proved the existence of Gaussian quadratures with multiple nodes and then Ghizzetti and Ossicini [8] established the following theorem.

Theorem A. *For any given set of odd multiplicities $\nu_1, \ldots, \nu_n$, there exists a unique quadrature formula of the form*

$$\int_a^b \mu(t) f(t) \mathrm{d}t \approx \sum_{j=1}^{n} \sum_{i=0}^{\nu_j - 1} a_{ji} f^{(i)}(x_j), \quad a \le x_1 < \cdots < x_n \le b, \tag{2.1}$$

of ADP $= \nu_1 + \cdots + \nu_n + n - 1$. *The nodes $x_1, \ldots, x_n$ of this quadrature are determined uniquely by the orthogonal property*

$$\int_a^b \mu(t) \prod_{k=1}^{n} (t - x_k)^{\nu_k} Q(t) \mathrm{d}t = 0, \quad \forall\, Q \in \pi_{n-1}(\mathbb{R}).$$

Quadratures of type (2.1) with equal multiplicities have been studied in [9], where Turàn proved Theorem A in the particular case when $\nu_1 = \cdots = \nu_n = \nu$, with ν being an odd number. The optimal nodes $\{x_j\}_{j=1}^{n}$ are characterized by the orthogonal property

$$\int_a^b \mu(t) \prod_{k=1}^{n} (t - x_k)^{\nu} Q(t) \mathrm{d}t = 0, \quad \forall\, Q \in \pi_{n-1}(\mathbb{R}).$$

In this case, the Gaussian quadrature is called Turàn quadrature of type ν. Further extension of Theorem A to the case of generalized Gaussian formulas with a sign changing weight was given in [10].

Next, we describe the connection between quadratures with multiple nodes and formulas of type (1.4). For the system of nodes $\mathbf{x} := (x_1, \ldots, x_n)$ with corresponding multiplicities $\bar{\nu} := (\nu_1, \ldots, \nu_n)$, we define the polynomials

$$\Lambda(t; \mathbf{x}) := \prod_{m=1}^{n} (t - x_m), \qquad \Lambda_j(t; \mathbf{x}) := \frac{\Lambda(t; \mathbf{x})}{(t - x_j)}, \qquad \Lambda^{\bar{\nu}}(t; \mathbf{x}) := \prod_{m=1}^{n} (t - x_m)^{\nu_m}$$

and set

$$x_j^{\nu_j} := (\underbrace{x_j, \ldots, x_j}_{\nu_j \text{ times}}), \quad j = 1, \ldots, n.$$

The following theorem reveals the relation between the standard quadratures and the quadratures for Fourier coefficients we are going to study.

Theorem 2.1. *For any given sets of multiplicities $\bar{\mu} = (\mu_1, \ldots, \mu_k)$ and $\bar{\nu} = (\nu_1, \ldots, \nu_n)$, and nodes $y_1 < \cdots < y_k$, $x_1 < \cdots < x_n$, there exists a quadrature formula of the form*

$$\int_a^b \mu(t) \Lambda^{\bar{\mu}}(t; \mathbf{y}) f(t) \mathrm{d}t \approx \sum_{j=1}^{n} \sum_{i=0}^{\nu_j - 1} c_{ji} f^{(i)}(x_j), \tag{2.2}$$

with ADP $= N$ *if and only if there exists a quadrature formula of the form*

$$\int_a^b \mu(t) f(t) \mathrm{d}t \approx \sum_{m=1}^{k} \sum_{\lambda=0}^{\mu_m - 1} b_{m\lambda} f^{(\lambda)}(y_m) + \sum_{j=1}^{n} \sum_{i=0}^{\nu_j - 1} a_{ji} f^{(i)}(x_j), \tag{2.3}$$

which has degree of precision $N + \mu_1 + \cdots + \mu_k$. In the case $y_m = x_j$ for some m and j, the corresponding terms in both sums combine in one term of the form

$$\sum_{\lambda=0}^{\mu_m + \nu_j - 1} d_{m\lambda} f^{(\lambda)}(y_m).$$

Proof. Assume that ADP (2.3) $= N + \mu_1 + \cdots + \mu_k$. Now apply (2.3) to the polynomial $\Lambda^{\bar{\mu}}(\cdot;\mathbf{y})f$, where $f \in \pi_N(\mathbb{R})$. The value of the first sum in (2.3) is 0 and we obtain

$$\int_a^b \mu(t)\Lambda^{\bar{\mu}}(t;\mathbf{y})f(t)\mathrm{d}t = \sum_{j=1}^{n}\sum_{i=0}^{\nu_j-1} a_{ji}\left[\Lambda^{\bar{\mu}}(t;\mathbf{y})f(t)\right]^{(i)}\Big|_{t=x_j}. \tag{2.4}$$

Since

$$\left[\Lambda^{\bar{\mu}}(t;\mathbf{y})f(t)\right]^{(i)} = \sum_{s=0}^{i}\binom{i}{s}\left[\Lambda^{\bar{\mu}}(t;\mathbf{y})\right]^{(s)} f^{(i-s)}(t),$$

equality (2.4) is a quadrature of form (2.2) which is exact for $f \in \pi_N(\mathbb{R})$.

Conversely, let us assume that ADP (2.2) $= N$. Observe that any sufficiently smooth function f has the representation

$$f(t) = H(t) + f[y_1^{\mu_1},\ldots,y_k^{\mu_k},t]\Lambda^{\bar{\mu}}(t;\mathbf{y}),$$

where H is the polynomial that interpolates f at the nodes $\mathbf{y} := (y_1,\ldots,y_k)$ with multiplicities $\bar{\mu} := (\mu_1,\ldots,\mu_k)$. We multiply both sides by the weight μ, integrate over $[a,b]$ and arrive at the formula

$$\int_a^b \mu(t)f(t)\mathrm{d}t = \int_a^b \mu(t)H(t)\mathrm{d}t + \int_a^b \mu(t)f[y_1^{\mu_1},\ldots,y_k^{\mu_k},t]\Lambda^{\bar{\mu}}(t;\mathbf{y})\mathrm{d}t.$$

The term $\int_a^b \mu(t)H(t)\mathrm{d}t$ is a sum like the first sum on the right side of (2.3). Notice that if f is a polynomial of degree $N + \mu_1 + \cdots + \mu_k$, then the divided difference $f[y_1^{\mu_1},\ldots,y_k^{\mu_k},\cdot]$ is a polynomial of degree at most N and we can compute the last integral exactly by (2.2). In conclusion, we obtain a formula of the form (2.3) that is exact for all polynomials of degree $N + \mu_1 + \cdots + \mu_k$. The proof is completed. □

It follows from the above proof that any formula of the form (2.2) generates a formula of the form (2.3), and vice versa. Surprisingly, the relation between (2.2) and (2.3) was used before only in one direction — to obtain from (2.3) a formula of type (2.2).

Note that we did not use in the proof of Theorem 2.1 that μ is a non-negative function. Thus the assertion holds for every integrable function μ.

A direct application of Theorem 2.1 for particular choices of the multiplicities $\bar{\mu}$ and $\bar{\nu}$ gives a characterization of the Gaussian quadratures for the Fourier coefficients.

Corollary 2.2. *Let $P_k(t) = (t-y_1)\cdots(t-y_k)$ and $y_i \neq x_j$ for all i and j. Formula (1.4), with $\nu_j, j = 1,\ldots,n$, odd multiplicities is Gaussian if and only if the polynomial $P_k\Lambda(\cdot;\mathbf{x})$ is orthogonal in $[a,b]$ with weight*

$$\mu(t)\prod_{j=1}^{n}(t-x_j)^{\nu_j-1}$$

to every polynomial of degree $n+k-1$.

Proof. Notice that $\sigma(P_k) = k$ and $\tau_i = 0$ for all i, since $y_i \neq x_j$ for all i and j. Thus, formula (1.4) is Gaussian if and only if it has ADP $= \nu_1 + \cdots + \nu_n + n + k - 1$. According to Theorem 2.1, ADP (1.4) $= e(\nu_1) + \cdots + e(\nu_n) + k - 1$ if and only if the formula

$$\int_a^b \mu(t)f(t)\mathrm{d}t \approx \sum_{m=1}^{k} b_{m0}f(y_m) + \sum_{j=1}^{n}\sum_{i=0}^{\nu_j-1} a_{ji}f^{(i)}(x_j)$$

has ADP $= 2k + \nu_1 + \cdots + \nu_n + n - 1$, i.e., if it is Gaussian. But, by Theorem A, the latter is true if and only if the polynomial $P_k(t)(t-x_1)\cdots(t-x_n)$ is orthogonal in $[a,b]$ with weight $\mu(t)\prod_{j=1}^{n}(t-x_j)^{\nu_j-1}$ to every polynomial of degree $n+k-1$. The proof is completed. □

Similar statement holds in the case when some of the nodes $y_1 < \cdots < y_k$ coincide with some of the nodes $x_1 < \cdots < x_n$. As illustration, we will only state the case when $y_1 = x_1$:

Let $P_k(t) = (t-x_1)(t-y_2)\cdots(t-y_k)$ and $y_i \neq x_j$ for $i = 2,\ldots,k$, and $j = 2,\ldots,n$. Formula (1.4) with odd multiplicities $\nu_j, j = 1,\ldots,n$, is Gaussian if and only if the polynomial $P_k\Lambda(\cdot;\mathbf{x})$ is orthogonal in $[a,b]$ with weight

$$\mu(t)\prod_{j=1}^{n}(t-x_j)^{\nu_j-1}$$

to every polynomial of degree $n+k-3$.

Note that Corollary 2.2 does not assert the existence of Gaussian quadratures for any weight and any fixed system of points $y_1 < \cdots < y_k$. It gives only a characterization. Later we shall return to the question of existence.

Next we fix the nodes and look for a polynomial P_k so that the resulting quadrature is Gaussian. The following is true.

Corollary 2.3. *For any given set of fixed nodes $x_1 < \cdots < x_n$ in $[a, b]$ and even multiplicities $\bar{\nu} = (\nu_1, \ldots, \nu_n)$, there exists a unique Gaussian quadrature formula of the form (1.4) with a certain polynomial $P_k(t) = (t - y_1) \cdots (t - y_k)$. P_k is characterized by the property that it is the polynomial of degree k, orthogonal in $[a, b]$ with weight $\mu(t)\Lambda^{\bar{\nu}}(t; \mathbf{x})$ to every polynomial of degree $k - 1$.*

Proof. It is a classical result that for the weight $\mu\Lambda^{\bar{\nu}}(\cdot; \mathbf{x}) \geq 0$ (all ν_i's are even) and the interval $[a, b]$, there exists a unique quadrature of the form

$$\int_a^b \mu(t)\Lambda^{\bar{\nu}}(t; \mathbf{x})f(t)\mathrm{d}t \approx \sum_{m=1}^{k} d_m f(y_m),$$

exact for all polynomials of degree $2k-1$. This is the Gauss formula, whose nodes $\{y_m\}_{m=1}^k$ are the zeros of the kth orthogonal polynomial on $[a, b]$ with weight $\mu\Lambda^{\bar{\nu}}(\cdot; \mathbf{x})$. Let us first assume that $y_m \neq x_i$ for all m and i. Then, by Theorem 2.1, there exists a unique quadrature of the form

$$\int_a^b \mu(t)f(t)\mathrm{d}t \approx \sum_{m=1}^{k} b_{m0}f(y_m) + \sum_{j=1}^{n}\sum_{i=0}^{\nu_j-1} a_{ji}f^{(i)}(x_j)$$

of ADP $= \nu_1 + \cdots + \nu_n + 2k - 1$. Again, by Theorem 2.1 the latter formula exists if and only if there is a quadrature of the form (1.4) of ADP $= \nu_1 + \cdots + \nu_n + k - 1$. This completes the proof, since $\sigma(P_k) = k$, $\tau_i = 0$, $i = 1, \ldots, n$ (here we consider the case when $y_m \neq x_i$ for all m and i) and for even ν_i, $e(\nu_i) = \nu_i$. The case when some of the nodes $y_1 < \cdots < y_k$ coincide with some of the $x_1 < \cdots < x_n$ can be treated similarly. □

We formulate below the important particular case when $k = n$ and $y_j = x_j$ for $j = 1, \ldots, n$.

Corollary 2.4. *For any given set of even multiplicities $\bar{\nu} = (\nu_1, \ldots, \nu_n)$, and weight μ, there exists a unique Gaussian quadrature (with ADP $= \nu_1 + \cdots + \nu_n + n - 1$) of the form*

$$\int_a^b \mu(t)\Lambda(t; \mathbf{x})f(t)\mathrm{d}t \approx \sum_{j=1}^{n}\sum_{i=0}^{\nu_j-1} c_{ji}f^{(i)}(x_j). \tag{2.5}$$

Its nodes $\{x_j\}_{j=1}^n$ are determined by the orthogonality property

$$\int_a^b \mu(t)\prod_{j=1}^{n}(t - x_j)^{\nu_j+1}Q(t)\mathrm{d}t = 0, \quad \forall\, Q \in \pi_{n-1}(\mathbb{R}).$$

Proof. Here $\sigma(\Lambda(t; \mathbf{x})) = n$, $\tau_i = 1$, $i = 1, \ldots, n$, and since the multiplicities ν_i are even,

$$M := \sum_{i=1}^{n} e(\nu_i - 1) + n - 1 = \sum_{i=1}^{n} \nu_i + n - 1.$$

Thus, by Theorem 2.1, formula (2.5) is Gaussian, i.e., it has ADP $= M$, if and only if there is a quadrature of the form

$$\int_a^b \mu(t)f(t)\mathrm{d}t \approx \sum_{j=1}^{n}\sum_{i=0}^{\nu_j} a_{ji}f^{(i)}(x_j)$$

of ADP $= M + n$. Now, by Theorem A, the last quadrature exists and is uniquely determined by the weight μ and the multiplicities $(\nu_1 + 1, \ldots, \nu_n + 1)$. The characterization of its nodes $\{x_j\}_{j=1}^n$ is given by Theorem A as well. □

Corollary 2.4 can be applied in the case when all multiplicities are even and equal to ν. Then the associated quadrature of type (2.1) is the Turàn quadrature of type $\nu + 1$, and the following particular case of the previous corollary holds.

Corollary 2.5. *For any even ν and weight μ, there exists a unique Gaussian quadrature (with ADP $= n\nu + n - 1$) of the form*

$$\int_a^b \mu(t)\Lambda(t; \mathbf{x})f(t)\mathrm{d}t \approx \sum_{j=1}^{n}\sum_{i=0}^{\nu-1} c_{ji}f^{(i)}(x_j). \tag{2.6}$$

Its nodes $\{x_j\}_{j=1}^n$ are the nodes of the Turàn quadrature of type $(\nu + 1)$ and $\operatorname{sign}\{c_{j,\nu-1}\} = (-1)^{n-j}$.

Proof. Since Corollary 2.5 follows directly from Corollary 2.4, it only remains to show that $\mathrm{sign}\{c_{j,\nu-1}\} = (-1)^{n-j}$. We consider the Turàn quadrature (2.1) of type $\nu+1$ with $\nu_j = \nu+1$ and apply it to $\Lambda(\cdot;\mathbf{x})Q$, where Q is a polynomial of degree $n\nu+n-1$. We have that

$$[\Lambda(t;\mathbf{x})Q(t)]^{(i)}(x_j) = i\Lambda_j(x_j;\mathbf{x})Q^{(i-1)}(x_j) + \sum_{\ell=0}^{i-2} d_{\ell j}Q^{(\ell)}(x_j), \quad i=2,\dots,\nu,$$

and therefore we obtain (2.6), where $c_{j,\nu-1} = \nu a_{j\nu}\Lambda_j(x_j;\mathbf{x})$. Since the coefficients $\{a_{ij}\}$ of Turàn's quadrature have the property (see [9]) $a_{ji} > 0$ for all j and even i, and

$$\mathrm{sign}\{\Lambda_j(x_j;\mathbf{x})\} = (-1)^{n-j},$$

we derive that $\mathrm{sign}\{c_{j,\nu-1}\} = (-1)^{n-j}$. $\square$

It is known that for every ν even, the nodes of the Turàn quadrature of type $(\nu+1)$ with weight $\mu(t) = (1-t^2)^{-1/2}$ on $[-1,1]$ are the zeros $\{\xi_j\}_{j=1}^n$ of the Tchebycheff polynomials of first kind T_n. This fact and an application of Corollary 2.5 with $[a,b] \equiv [-1,1]$ and $\mu(t) = (1-t^2)^{-1/2}$ gives the Gaussian formula

$$\int_{-1}^{1} \frac{T_n(t)}{\sqrt{1-t^2}} f(t)\mathrm{d}x \approx \sum_{j=1}^{n}\sum_{i=0}^{\nu-1} c_{ji}f^{(i)}(\xi_j)$$

with ADP $= n\nu+n-1$. The coefficients c_{ji} can be found from the coefficients of the corresponding Turàn quadrature.

In the case $\nu = 2$, we arrive at a formula based on the values of the integrand f and its derivative at the zeros of T_n. A calculation of the coefficients of $f(\xi_k)$, $k=1,\dots,n$, in the resulting quadrature formula shows that they are equal to zero and thus one obtains the Micchelli–Rivlin quadrature (1.2) from [1] which is exact for $\pi_{3n-1}(\mathbb{R})$.

Corollary 2.3 gives a positive answer to the question of existence and uniqueness of Gaussian formulas of type (1.4) if we fix the nodes $\{x_j\}_{j=1}^n$ and their even multiplicities $(\nu_1,\dots,\nu_n)$. An affirmative answer has also the problem of existence and uniqueness of Gaussian quadratures of type (1.4) (see Corollary 2.4) where $k=n$ and the polynomial P_n has exactly n real zeros which are the nodes $\{x_j\}_{j=1}^n$.

Now let us return to the question of existence and uniqueness of a Gaussian quadrature of type (1.4) for any fixed polynomial $P_k(t) = (t-y_1)\cdots(t-y_k)$ and prescribed multiplicities $(\nu_1,\dots,\nu_n)$. This question does not have a definitive answer, as it can be seen from the following examples.

First, we show that if $[a,b] \equiv [-1,1]$, $\mu(t) \equiv 1$, $n=1$, $\nu_1 = 2$, $k=1$, and $P_1(t) = t-2/3$, there is no Gaussian quadrature of type (1.4). Namely, there is no $x_1 \in [-1,1]$ such that the formula

$$\int_{-1}^{1}\left(t-\frac{2}{3}\right)f(t)\mathrm{d}t \approx a_{10}f(x_1) + a_{11}f'(x_1),$$

is exact for all polynomials of degree 2. Indeed, this requirement leads to solving for x_1 the equation $12x_1^2+12x_1+4=0$ which does not have real roots. Next, one can easily show that if $[a,b] \equiv [-1,1]$, $\mu(t) \equiv 1$, $n=1$, $\nu_1 = 2$, $k=1$, and $P_1(t) = t-11/20$, both formulas

$$\int_{-1}^{1}\left(t-\frac{11}{20}\right)f(t)\mathrm{d}t \approx -\frac{11}{10}f\left(\frac{\sqrt{37}-20}{33}\right) + \frac{\sqrt{37}}{30}f'\left(\frac{\sqrt{37}-20}{33}\right)$$

and

$$\int_{-1}^{1}\left(t-\frac{11}{20}\right)f(t)\mathrm{d}t \approx -\frac{11}{10}f\left(\frac{-\sqrt{37}-20}{33}\right) - \frac{\sqrt{37}}{30}f'\left(\frac{-\sqrt{37}-20}{33}\right)$$

are Gaussian, i.e., they are exact for polynomials of degree 2.

As shown by the examples above, the existence (and uniqueness) of Gaussian quadratures (1.4) is not secured for any fixed set of a polynomial P_k and multiplicities $\bar{\nu} := (\nu_1,\dots,\nu_n)$. Actually, Corollary 2.2 describes all triplets $(\mu, P_k, \bar{\nu})$, with $\bar{\nu}$ being a set of odd multiplicities, for which such extremal quadratures exist. Indeed, the point $\mathbf{x} := (x_1,\dots,x_n) \in [a,b]^n$, $x_j \neq y_i$ for all j and i, is *Gaussian* for the triplet $(\mu, P_k, \bar{\nu})$, that is, there is a Gaussian formula of type (1.4), if the polynomial

$$P_k(t)(t-x_1)\cdots(t-x_n)$$

is orthogonal in $[a,b]$ with weight

$$\mu(t)\prod_{j=1}^{n}(t-x_j)^{\nu_j-1}$$

to every polynomial of degree $n+k-1$. But this means that the zeros $y_1,\dots,y_k$ of P_k, together with the points $x_1,\dots,x_n$ are the nodes of the Gaussian quadrature formula associated with the weight μ and the multiplicities $\mu_i = 1$ for $i=1,\dots,k$, and odd $\nu_1, \ldots, \nu_n$. Such a formula always exists (and is unique for any preassigned order of the multiplicities $\{\mu_i\}$ and $\{\nu_j\}$, see Theorem A). Therefore, for any chosen positions $i_1 < \cdots < i_k$ among $(1, 2, \ldots, n+k)$ there is a unique Gaussian quadrature formula of the type

$$\int_a^b \mu(t) f(t) \mathrm{d}t \approx \sum_{j=1}^{n+k} \sum_{i=0}^{s_j-1} a_{ji} f^{(i)}(t_j),$$

where $s_{i_m} = 1$ for $m = 1, \ldots, k$, and the rest of the multiplicities s_j coincide with $\nu_1, \ldots, \nu_n$, given in a preassigned order. Clearly, if we choose the polynomial P_k, such that its zeros y_m, $m = 1, \ldots, k$, are

$$y_m = t_{i_m}, \quad m = 1, \ldots, k,$$

then there exists a Gaussian quadrature of type (1.4). Only such choices of P_k admit a Gaussian formula. Similar argument holds if we have that $x_j = y_i$ for some i and j.

In [5], for every $s > 0$, Micchelli and Sharma constructed a formula of the form

$$\int_{-1}^1 \frac{1}{\sqrt{1-t^2}} T_n(t) f(t) \mathrm{d}t \approx \sum_{j=0}^{s} \left[A_j f^{(j)}(-1) + B_j f^{(j)}(1)\right] + \sum_{j=1}^{n-1} \sum_{i=0}^{2s} a_{ji} f^{(i)}(x_j), \tag{2.7}$$

with ADP (2.7) $= (2s+3)n - 1$, which has the highest possible precision. Since the nodes of their formula are located at the extremal points $-1, \tilde{\eta}_1, \ldots, \tilde{\eta}_{n-1}, 1$, of the Tchebycheff polynomial T_n (note that $\{\tilde{\eta}_j\}_{j=1}^{n-1}$ are also the zeros of the Tchebycheff polynomial of second kind U_{n-1}), it can be considered as an extension of the simple node formula

$$\frac{2}{\pi} \int_{-1}^1 \frac{1}{\sqrt{1-t^2}} T_n(t) f(t) \mathrm{d}t \approx 2^{1-n} f[-1, \tilde{\eta}_1, \ldots, \tilde{\eta}_{n-1}, 1]$$

of ADP $= 3n - 1$, established earlier in [1]. It was not known whether the Micchelli–Sharma multiple node quadrature is unique, although it has the highest degree of precision. We can derive here the uniqueness from our characterization results.

Theorem 2.6. *Let*

$$\mathcal{M}_j[f] := f[(-1)^\sigma, \tilde{\eta}_1^j, \ldots, \tilde{\eta}_{n-1}^j, 1^\sigma], \quad \sigma := \left[\frac{j+1}{2}\right].$$

The Micchelli–Sharma quadrature formula

$$\frac{2}{\pi} \int_{-1}^1 \frac{1}{\sqrt{1-t^2}} T_n(t) f(t) \mathrm{d}t \approx 2^{1-n} \left\{ \mathcal{M}_1[f] + 2 \sum_{j=1}^{s} \frac{(-1)^j j a_j}{j+1} \mathcal{M}_{2j+1}[f] \right\},$$

where a_j are defined by their generating function

$$\sum_{j=0}^{\infty} j a_j t^j = \frac{1}{2}[(1 - 4^{-n+1} t)^{-1/2} - 1],$$

is the unique formula of the form (2.7) of highest ADP.

Proof. Assume that a quadrature of form (2.7) has ADP $= (2s+3)n - 1$ and $x_j \neq \xi_i$ for all j and i. Then, by Theorem 2.1, the quadrature

$$\int_{-1}^1 \frac{1}{\sqrt{1-t^2}} f(t) \mathrm{d}t \approx \sum_{j=0}^{s} \left[A_j f^{(j)}(-1) + B_j f^{(j)}(1)\right] + \sum_{j=1}^{n-1} \sum_{i=0}^{2s} a_{ji} f^{(i)}(x_j) + \sum_{j=1}^{n} \alpha_j f(\xi_j)$$

will be exact for all polynomials of degree $(2s+4)n - 1$ and consequently, again by Theorem 2.1, a quadrature of the form

$$\int_{-1}^1 (1-t^2)^{s+1} \frac{1}{\sqrt{1-t^2}} f(t) \mathrm{d}t \approx \sum_{j=1}^{n-1} \sum_{i=0}^{2s} a_{ji} f^{(i)}(x_j) + \sum_{j=1}^{n} \alpha_j f(\xi_j)$$

will integrate exactly all polynomials of degree $(2s+4)n - 1 - (2s+2) = (2s+2)(n-1) + 2n - 1$. This formula is based on n interior simple nodes $\xi_1, \ldots, \xi_n$, and $(n-1)$ nodes $x_1, \ldots, x_{n-1}$, each of odd multiplicity $2s+1$. Such formula can attain the highest possible ADP, equal to

$$\mathrm{ADP} = 2n + (2s+2)(n-1) - 1,$$

only in the case when all nodes are Gaussian. But by Theorem A, there exists only one such quadrature. Thus, it should coincide with the formula corresponding (via Theorem 2.1) to the Micchelli–Sharma quadrature. Similar argument holds when some of the points in $\{x_j\}_{j=1}^{n-1}$ coincide with some of the points $\{\xi_i\}_{i=1}^{n}$. The proof is complete. $\square$

Note that applying Theorem 2.1 once more to the last quadrature formula, we arrive at the following conclusion.

Corollary 2.7. *There exists a unique Gaussian quadrature formula (for the Fourier–Tchebycheff coefficients) of the form*

$$\int_{-1}^{1}(1-t^2)^{s+1/2}T_n(t)f(t)\mathrm{d}t \approx \sum_{j=1}^{n-1}\sum_{i=0}^{2s} a_{ji}f^{(i)}(x_j)$$

of ADP $=(2s+2)(n-1)-1$. *The nodes of this extremal formula are located at the zeros* $\tilde{\eta}_1,\ldots,\tilde{\eta}_{n-1}$ *of* U_{n-1} *and the coefficients can be given explicitly via the coefficients of the Micchelli–Sharma quadrature.*

3. Quadratures with simple nodes

In this section, we discuss quadratures with simple nodes, i.e., the information used to recover the integral consists of function values only. In this case, we give explicit expressions for the extremal nodes and the coefficients for some widely used weight functions.

3.1. Quadratures with free simple nodes

Here, we consider formulas of the type

$$\int_a^b \mu(t)P_k(t)f(t)\mathrm{d}t \approx \sum_{j=1}^{n} a_j f(x_j), \quad P_k(t)=(t-y_1)\ldots(t-y_k), \tag{3.1}$$

where $\mathbf{y}:=(y_1,\ldots,y_k)$, is a given set of points and $y_i \neq x_j$ for all i and j. We study the problem of characterization of those $\mathbf{y}$ for which the quadrature has maximal algebraic degree of precision. Clearly, for any choice of the parameters $\{a_j\}_{j=1}^n$ and $\{x_j\}_{j=1}^n$,

ADP (3.1) $< 2n+k$,

since it is not exact for the polynomial $P_k\Lambda^2(\cdot;\mathbf{x}) \in \pi_{2n+k}(\mathbb{R})$. The following version of Corollary 2.2 gives a complete characterization of the nodes $\mathbf{y}$ for which there exists a quadrature formula (3.1) of maximal ADP, equal to $2n+k-1$.

Theorem 3.1. *The quadrature formula* (3.1) *has* ADP $=2n+k-1$ *if and only if the nodes* $y_1,\ldots,y_k,x_1,\ldots,x_n$, *are the zeros of the polynomial* S_{n+k} *which is orthogonal in* $[a,b]$ *with weight* μ *to every polynomial of degree* $k+n-1$. *Moreover, the coefficients* $\{a_j\}_{j=1}^n$ *of the extremal quadrature are given by* $a_j=\alpha_jP_k(x_j)$, $j=1,\ldots,n$, *where* $\{\alpha_j\}_{j=1}^n$ *are the coefficients corresponding to the nodes* $x_1,\ldots,x_n$ *in the Gauss quadrature formula with* $n+k$ *nodes.*

Proof. Let us consider the Gaussian quadrature

$$\int_a^b \mu(t)g(t)\mathrm{d}t \approx \sum_{j=1}^{n}\alpha_j g(x_j)+\sum_{j=1}^{k}\beta_j g(y_j), \tag{3.2}$$

on $[a,b]$ with weight μ. Its nodes $x_1,\ldots,x_n,y_1,\ldots,y_k$, are the zeros of the polynomial of degree $n+k$ that is orthogonal to $\pi_{k+n-1}(\mathbb{R})$ on $[a,b]$ with weight μ. Since ADP (3.2) $=2(n+k)-1$, it should be exact for every polynomial of the form $g=P_kf$, where $f\in\pi_{2n+k-1}(\mathbb{R})$ and $P_k(t)=(t-y_1)\ldots(t-y_k)$. This yields

$$\int_a^b \mu(t)P_k(t)f(t)\mathrm{d}t=\sum_{j=1}^{n}\alpha_jP_k(x_j)f(x_j), \quad \text{for every } f\in\pi_{2n+k-1}(\mathbb{R}).$$

Thus, we derive a formula of type (3.1) with nodes $\{x_j\}_{j=1}^n$ and coefficients $a_j=\alpha_jP_k(x_j)$ of highest ADP.

Now assume that the quadrature (3.1) has ADP $=2n+k-1$. Consider the polynomial $S_{k+n}(t):=P_k(t)\Lambda(t;\mathbf{x})$. For every polynomial $Q\in\pi_{k+n-1}(\mathbb{R})$ we have

$$\int_a^b \mu(t)S_{k+n}(t)Q(t)\mathrm{d}t=\int_a^b \mu(t)P_k(t)\Lambda(t;\mathbf{x})Q(t)\mathrm{d}t=\sum_{j=1}^{n}a_j\Lambda(x_j;\mathbf{x})Q(x_j)=0.$$

In other words, the polynomial S_{k+n} is orthogonal to every polynomial from $\pi_{k+n-1}(\mathbb{R})$ on $[a,b]$ with weight μ. It remains to show that the coefficients a_j are related to the Gaussian coefficients α_j by $a_j=\alpha_jP_k(x_j)$. To show this we apply quadrature (3.1) and the Gaussian quadrature (3.2) to the polynomials $\Lambda_j(t;\mathbf{x})$ and $P_k(t)\Lambda_j(t;\mathbf{x})$, respectively, and derive

$$\int_a^b \mu(t)P_k(t)\Lambda_j(t;\mathbf{x})\mathrm{d}t=a_j\Lambda_j(x_j;\mathbf{x});$$

$$\int_a^b \mu(t)P_k(t)\Lambda_j(t;\mathbf{x})\mathrm{d}t=\alpha_jP_k(x_j)\Lambda_j(x_j;\mathbf{x}).$$

Comparing the above expressions, we obtain the wanted relation and the proof is completed. □

Theorem 3.1 characterizes completely the Gaussian quadrature of type (3.1) in the case of simple nodes. It shows, that given a polynomial P_k, it is not always possible to construct a Gaussian quadrature of type (3.1). Necessary and sufficient condition for such a formula to exist is the fact that the zeros of P_k, $\{y_j\}_{j=1}^k$, have to be among the zeros of the $(n+k)$th polynomial (call it S_{n+k}), orthogonal on $[a, b]$ with weight μ. Therefore:

If P_k divides S_{n+k}, then there exists a unique quadrature of the form (3.1) of highest ADP, equal to $2n+k-1$.

We emphasize this remark, since it is related to the question of uniqueness of the Gaussian quadrature for Fourier coefficients.

Next, we will give some examples of Gaussian quadratures. Let us first recall that

$$T_n(t) = \cos n\theta, \qquad U_n(t) = \frac{\sin(n+1)\theta}{\sin\theta}, \quad \text{where } t = \cos\theta,$$

are the Tchebycheff polynomials of first and second kind, respectively. It is a well-known fact that the nodes of the Gaussian quadrature formula of the form

$$\int_{-1}^{1} \sqrt{1-t^2} f(t)\mathrm{d}t \approx \sum_{j=1}^{n} \beta_j f(\eta_j) + \sum_{j=1}^{n+1} \alpha_j f(\xi_j), \tag{3.3}$$

are the zeros of U_{2n+1}. Using the identity

$$U_{2n+1}(t) = 2T_{n+1}(t)U_n(t),$$

we conclude that the nodes of (3.3) coincide with the zeros $\eta_1, \ldots, \eta_n$ of U_n and the zeros $\xi_1, \ldots, \xi_{n+1}$ of T_{n+1}. The explicit form of (3.3) is well known:

$$\int_{-1}^{1} \sqrt{1-t^2} f(t)\mathrm{d}t \approx \frac{\pi}{2n+2} \sum_{j=1}^{n} \left(\sin \frac{j\pi}{n+1}\right)^2 f\left(\cos \frac{j\pi}{n+1}\right) + \frac{\pi}{2n+2} \sum_{k=1}^{n+1} \left(\sin \frac{(2k-1)\pi}{2n+2}\right)^2 f\left(\cos \frac{(2k-1)\pi}{2n+2}\right).$$

The following corollaries are a simple consequence of Theorem 3.1.

Corollary 3.2. *The quadrature formula*

$$\int_{-1}^{1} \sqrt{1-t^2} U_n(t) f(t)\mathrm{d}t \approx \sum_{k=1}^{n+1} a_k f(\xi_k),$$

with

$$a_k = \frac{\pi}{2n+2} \left(\sin \frac{(2k-1)\pi}{2n+2}\right)^2 U_n(\xi_k) = (-1)^{k-1} \frac{\pi}{2n+2} \sin \frac{(2k-1)\pi}{2n+2}$$

is the unique formula of highest ADP (equal to $3n+1$) among all formulas of this type with $n+1$ nodes. Here $\xi_k = \cos \frac{(2k-1)\pi}{2n+2}$, $k = 1, \ldots, n+1$, are the zeros of the Tchebycheff polynomial of first kind T_{n+1}.

Corollary 3.3. *The quadrature formula*

$$\int_{-1}^{1} \sqrt{1-t^2} T_{n+1}(t) f(t)\mathrm{d}t \approx \sum_{j=1}^{n} b_j f(\eta_j),$$

with

$$b_j = (-1)^j \frac{\pi}{2n+2} \left(\sin \frac{j\pi}{n+1}\right)^2 \quad \text{and} \quad \eta_j = \cos \frac{j\pi}{n+1}$$

is the unique formula of highest ADP (equal to $3n$) among all formulas of this type with n nodes.

3.2. Connection to Gauss–Kronrod quadratures

In this section we discuss the relation between the Gauss–Kronrod quadratures and quadratures of type (3.1) when the polynomial P_k is fixed.

To improve the numerical results obtained by the application of a certain quadrature formula, Kronrod [4] suggested the following procedure. Assume that we start with a quadrature

$$\int_a^b \mu(t)f(t)\mathrm{d}t \approx \sum_{j=1}^{k} a_j f(y_j). \tag{3.4}$$

Keeping $\{y_j\}_{j=1}^k$ fixed, can we choose another set of n nodes $x_1 < \cdots < x_n$ so that the quadrature based on the nodes $\{y_j\}_{j=1}^k$ and $\{x_j\}_{j=1}^n$ has a maximal ADP. In other words, we are looking for a quadrature of the form

$$\int_a^b \mu(t)f(t)\mathrm{d}t \approx \sum_{j=1}^{k} \alpha_j f(y_j) + \sum_{j=1}^{n} \beta_j f(x_j) \tag{3.5}$$

with fixed $\{y_j\}_{j=1}^k$ which has maximal ADP. Note that here $y_j \neq x_i$ for all i and j.

The following lemma gives the correspondence between quadratures (3.5) and quadratures of type (3.1) with fixed polynomial P_k.

Lemma 3.4. *Quadrature formula (3.5) with fixed nodes $\{y_j\}_{j=1}^k$ has ADP $= N + k$ if and only if the quadrature*

$$\int_a^b \mu(t)P_k(t)f(t)\mathrm{d}t \approx \sum_{j=1}^{n} b_j f(x_j), \quad P_k(t) = (t-y_1)\cdots(t-y_k), \tag{3.6}$$

has ADP $= N$. Moreover the relationship between $\{\alpha_j\}_{j=1}^k$, $\{\beta_j\}_{j=1}^n$, and $\{b_j\}_{j=1}^n$ is given by

$$\sum_{j=1}^{k} \alpha_j f(y_j) + \sum_{j=1}^{n} \beta_j f(x_j) = \sum_{j=1}^{k} \gamma_j f(y_j) + \sum_{j=1}^{n} b_j f[y_1,\ldots,y_k,x_j],$$

where

$$\gamma_j = \frac{1}{P_k'(y_j)} \int_a^b \mu(t)\frac{P_k(t)}{t-y_j}\mathrm{d}t.$$

Proof. The equivalence of the two quadratures is a direct consequence of Theorem 2.1. The relation between the coefficients of these quadratures follows from the proof of Theorem 2.1 which gives that formula (3.5) can be written in the form

$$\int_a^b \mu(t)f(t)\mathrm{d}t \approx \sum_{j=1}^{k} \gamma_j f(y_j) + \sum_{j=1}^{n} b_j f[y_1,\ldots,y_k,x_j],$$

$$\gamma_j = \frac{1}{\Lambda_j(y_j;\mathbf{y})} \int_a^b \mu(t)\Lambda_j(t;\mathbf{y})\mathrm{d}t,$$

where $\{b_j\}_{j=1}^n$ are the coefficients from (3.6). $\square$

The question is whether one can choose the nodes $\{x_j\}_{j=1}^n$ so that N is as high as possible. Clearly, the maximal N satisfies $N \geq n-1$, since we can always choose an interpolation quadrature (3.6) based on the set of nodes $\{x_j\}_{j=1}^n$.

Let us introduce the polynomial

$$E_n(t) := (t-x_1)\ldots(t-x_n).$$

According to Theorem 2.1, quadrature (3.5) is exact for all polynomials of degree $n+k+m$ if and only if quadrature (3.6) is exact for all polynomials of degree $n+m$. It can be easily seen that the latter is true if and only if

$$\int_a^b \mu(t)(t-y_1)\ldots(t-y_k)E_n(t)Q(t)\mathrm{d}t = 0, \quad \text{for all } Q \in \pi_m(\mathbb{R}).$$

In the case when $n = k+1$, and quadrature (3.4) is the Gauss quadrature formula on $[a,b]$, associated with the weight μ, the quadrature (3.5) of ADP $\geq 2k$ is known as Gauss–Kronrod quadrature.

Our observation above, applied in the case when (3.4) is the Gaussian quadrature on $[a,b]$ with weight μ, $n = k+1$, and $m = k$ gives the following classical result on Gauss–Kronrod quadratures (see [11,12]).

Corollary 3.5. *Let (3.4) be the Gauss quadrature on $[a,b]$ with weight μ. Then quadrature (3.5) with $n = k+1$ is exact for all polynomials of degree $3k+1$ if and only if*

$$\int_a^b \mu(t)(t-y_1)\ldots(t-y_k)E_{k+1}(t)Q(t)\mathrm{d}t = 0, \quad \textit{for all } Q \in \pi_k(\mathbb{R}).$$

For a given weight μ, the polynomial E_{k+1} that satisfies these orthogonality relations is called the Stieltjes polynomial. The problem of existence of such a polynomial for a particular weight goes back to Stieltjes. Szegö proved its existence in the case of constant weight and raised some still open questions concerning existence for other classical weights.

Lemma 3.4 shows that the question of constructing formulas of type (3.1) of highest ADP with $k+1$ nodes in the special case when P_k is fixed to be the kth orthogonal polynomial on $[a,b]$ with weight μ becomes a question of constructing Gauss–Kronrod formulas on $[a,b]$ with weight μ. More precisely, the formula

$$\int_a^b \mu(t)f(t)\mathrm{d}t \approx \sum_{j=1}^{k} \alpha_j f(y_j) + \sum_{j=1}^{k+1} \beta_j f(x_j)$$

is a Gauss–Kronrod formula with ADP $= N$ if and only if the quadrature

$$\int_a^b \mu(t)(t-y_1)\cdots(t-y_k)f(t)\mathrm{d}t \approx \sum_{j=1}^{k+1} b_j f(x_j)$$

has ADP $= N-k$. The coefficients are given by the relation $b_j = \beta_j P_k(x_j), j = 1,\dots,k+1$.

For results on existence, nonexistence, and construction of Kronrod type extensions of the Gaussian rule for the integral $\int_a^b \mu(t)f(t)\mathrm{d}t$ for particular weight functions μ, we refer the reader to [13–15,12,16–19] and the references therein.

As discussed at the beginning of this paper, we have that $N \le 2n+k-1$, and the quadrature is called Gaussian, if $N = 2n+k-1$. Theorem 3.1 shows that the cases of Gaussian quadratures (3.6) happen only if the fixed $\{y_j\}_{j=1}^k$ are among the zeros of S_{n+k}, the $(n+k)$th polynomial, orthogonal on $[a,b]$ with weight μ. In the case of Gauss–Kronrod formulas, the nodes $\{y_j\}_{j=1}^k$ are the nodes of the Gauss quadrature on $[a,b]$ with weight μ, i.e. they are the zeros of S_k. Therefore, the ADP (3.5) $= 2n+2k-1$ if and only if $S_{n+k} = S_k R_n$ for some $R_n \in \pi_n(\mathbb{R})$. Example of Gauss–Kronrod formula (3.5) with ADP $= 4k+1$ is (3.3).

In the light of this discussion, the question of uniqueness of the Micchelli–Rivlin quadrature formula (1.2) amid all formulas of type (1.3) reduces to the question: Do there exist points $x_1 < \cdots < x_n$, other than $\{\xi_j\}_{j=1}^n$, such that the n nodes Gauss quadrature with Tchebycheff weight $1/\sqrt{1-t^2}$ can be extended to the form

$$\int_{-1}^1 \frac{f(t)}{\sqrt{1-t^2}}\mathrm{d}t \approx \sum_{j=1}^n \alpha_j f(\xi_j) + \sum_{j=1}^n [\beta_j f(x_j) + \gamma_j f'(x_j)]$$

of ADP = $4n-1$? Or, equivalently: Does there exists a weight of the form

$$\mu(t) = \frac{(t-x_1)^2\cdots(t-x_n)^2}{\sqrt{1-t^2}},$$

with $\{x_j\}_{j=1}^n \ne \{\xi_j\}_{j=1}^n$, so that T_n is orthogonal with respect to this weight to all polynomials of degree $n-1$? As we mentioned already, the answer is "no". But a similar multiple node version of this question is still open. Namely: Given any natural number ν, does there exists a weight of the form

$$\mu(t) = \frac{(t-x_1)^{2\nu}\cdots(t-x_n)^{2\nu}}{\sqrt{1-t^2}},$$

with $\{x_j\}_{j=1}^n \ne \{\xi_j\}_{j=1}^n$, so that T_n is orthogonal with respect to this weight to all polynomials of degree $n-1$? The answer of this question will settle a long time open problem about the uniqueness of a multiple node Gaussian formula for the Fourier–Tchebycheff coefficients (see [5]).

It would be also interesting to study the existence of Gauss–Kronrod extensions of the form

$$\int_{-1}^1 \mu(t)f(t)\mathrm{d}t \approx \sum_{j=1}^n \alpha_j f(\tau_j) + \sum_{j=1}^n [\beta_j f(x_j) + \gamma_j f'(x_j)]$$

of ADP $= 4n-1$ for other weights, say for the constant weight.

3.3. Quadratures with fixed simple nodes

Here, we shall discuss quadratures of the form

$$\int_a^b \mu(t)P_k(t)f(t)\mathrm{d}t \approx \sum_{j=1}^k b_j f(y_j) + \sum_{j=1}^n a_j f(x_j), \quad P_k(t) = (t-y_1)\dots(t-y_k). \tag{3.7}$$

The nodes $y_1,\dots,y_k$ are fixed and we look for n other points $x_1,\dots,x_n$, to obtain highest possible ADP. Clearly, if $n=0$ the maximal ADP of such a formula will be $k-1$, since the formula will not be exact for P_k. In this case any interpolatory formula based on the nodes $\{y_j\}_{j=1}^k$ is Gaussian. The following theorem is true.

Theorem 3.6. *For any given set of distinct points $\mathbf{y} := (y_1, \dots, y_k)$ there exists a unique quadrature formula of the form (3.7) of highest ADP (which is $2n + k - 1$). The nodes $\{x_j\}_{j=1}^n$ and the numbers $a_j P_k(x_j)$, $j = 1, \dots, n$, coincide with the nodes and the coefficients of the Gauss quadrature formula with n nodes on the interval $[a, b]$ with the weight μP_k^2, and therefore $a_j P_k(x_j)$, $j = 1, \dots, n$, are positive.*

Proof. Let L_{k-1} be the Lagrange interpolating polynomial of degree $k - 1$ for the function f and nodes $\mathbf{y}$. Then, by Newton's formula,

$$f(t) = L_{k-1}(t) + f[y_1, \dots, y_k, t]P_k(t).$$

We multiply this identity by μP_k and integrate over $[a, b]$ to obtain

$$\int_a^b \mu(t)P_k(t)f(t)\mathrm{d}t = \int_a^b \mu(t)P_k(t)L_{k-1}(t)\mathrm{d}t + \int_a^b \mu(t)P_k^2(t)f[y_1, \dots, y_k, t]\mathrm{d}t. \tag{3.8}$$

Using the Gauss quadrature formula associated with the weight μ and the interval $[a, b]$ (with certain nodes $z_1, \dots, z_k$ and coefficients $\gamma_1, \dots, \gamma_k$), we compute

$$\int_a^b \mu(t)P_k(t)L_{k-1}(t)\mathrm{d}t = \sum_{j=1}^k \gamma_j P_k(z_j)L_{k-1}(z_j) = \sum_{j=1}^k c_j f(y_j). \tag{3.9}$$

The last equality holds with some $\{c_j\}_{j=1}^k$ since the values $L_{k-1}(z_j)$ can be expressed in terms of $f(y_1), \dots, f(y_k)$. Note that the above evaluation of the integral holds for every function f, and hence for every polynomial of degree $2n + k - 1$. Next, we compute the second integral in (3.8) by the Gauss quadrature formula associated with the weight μP_k^2 (with some nodes $x_1, \dots, x_n$ and coefficients $\alpha_1, \dots, \alpha_n$). Note that for every polynomial f of degree $2n + k - 1$ the divided difference $f[y_1, \dots, y_k, t]$ is a polynomial of degree $2n - 1$. Thus the next evaluation is exact for every $f \in \pi_{2n+k-1}(\mathbb{R})$ and we have

$$\int_a^b \mu(t)P_k^2(t)f[y_1, \dots, y_k, t]\mathrm{d}t = \sum_{j=1}^n \alpha_j f[y_1, \dots, y_k, x_j] = \sum_{j=1}^n \frac{\alpha_j}{P_k(x_j)} f(x_j) + \sum_{j=1}^k \tilde{\alpha}_j f(y_j)$$

with some coefficients $\tilde{\alpha}_j$. Now substituting this equality and (3.9) into (3.8), we obtain a formula of the form (3.7) with ADP $= 2n + k - 1$.

The uniqueness follows easily. Indeed, assume that (3.7) has ADP $= 2n + k - 1$. Then, applying the formula to $P_k Q$, we get the equality

$$\int_a^b \mu(t)P_k^2(t)Q(t)\mathrm{d}t = \sum_{j=1}^n a_j P_k(x_j)Q(x_j)$$

for every $Q \in \pi_{2n-1}(\mathbb{R})$, and hence the coefficients $\{a_j P_k(x_j)\}_{j=1}^n$ and the nodes $\{x_j\}_{j=1}^n$ are uniquely characterized as parameters of a Gaussian quadrature. The coefficients $\{b_j\}_{j=1}^k$ are uniquely determined by the condition that the formula is of interpolatory type and has nodes at $\{y_j\}_{j=1}^k$ and $\{x_j\}_{j=1}^n$. The proof is completed. $\square$

Theorem 3.6 shows that one can improve the precision of the quadrature

$$\int_a^b \mu(t)P_k(t)f(t)\mathrm{d}t \approx \sum_{j=1}^k \beta_j f(y_j), \quad P_k(t) = (t - y_1)\dots(t - y_k),$$

following the strategy of Kronrod, namely by adding additional nodes $\{x_j\}_{j=1}^n$. One can achieve the highest possible precision $2n + k - 1$ only by adding specific nodes, the nodes of the Gauss quadrature on $[a, b]$ with weight μP_k^2.

The nodes $\{y_j\}_{j=1}^k$ of the starting formula do not have any impact on the accuracy of the resulting formula (3.7). However, we can select $\{y_j\}_{j=1}^k$ to be the zeros of the kth orthogonal polynomial on $[a, b]$ with weight μ. Then the following corollary is a particular application of Theorem 3.6.

Corollary 3.7. *Let $P_k(t) = (t - y_1)\dots(t - y_k)$ be the kth orthogonal polynomial on $[a, b]$ with weight μ. Then there exists a unique Gaussian quadrature (with ADP $= 2n + k - 1$) of type (3.7). This formula is*

$$\int_a^b \mu(t)P_k(t)f(t)\mathrm{d}t \approx \sum_{j=1}^n \alpha_j f[y_1, \dots, y_k, x_j],$$

where $\{\alpha_j\}_{j=1}^k$ are the weights and $\{x_j\}_{j=1}^n$ are the nodes of the Gauss formula on $[a, b]$ with weight μP_k^2,

$$\int_a^b \mu(t)P_k^2(t)f(t)\mathrm{d}t \approx \sum_{j=1}^n \alpha_j f(x_j).$$

Proof. The proof follows the argument in Theorem 3.6 and the fact that the value of the first integral in (3.8), $\int_a^b \mu(t)P_k(t)L_{k-1}(t)\mathrm{d}t = 0$, because of the orthogonality of P_k. This completes the proof. $\square$

Next, we give some examples of Gaussian formulas of type (3.7). Before going further, we need the following lemma.

Lemma 3.8. *For every integer $s \geq 0$, the quadrature*

$$\int_{-1}^{1} (1-t^2)^{s+\frac{1}{2}} U_n^{2s+2}(t) f(t)\mathrm{d}t \approx \frac{\pi}{2^{2s+2}n}\binom{2s+2}{s+1}\sum_{j=1}^{n} f(\xi_j), \tag{3.10}$$

where U_n is the Tchebycheff polynomial of second kind and $\{\xi_j\}_{j=1}^n$ are the zeros of the Tchebycheff polynomial of first kind T_n is exact for all polynomials in $\pi_{2n-1}(\mathbb{R})$.

Proof. The proof follows the argument from Lemma 2 in [6]. First, we need to show that T_n is orthogonal to $\pi_{n-1}(\mathbb{R})$ on $[-1, 1]$ with weight $(1-t^2)^{s+\frac{1}{2}} U_n^{2s+2}(t)$. Let Q be an arbitrary polynomial from $\pi_{n-1}(\mathbb{R})$. We use the relation (see Lemma 3.2 in [20])

$$\begin{aligned}(1-t^2)^{s+\frac{1}{2}} U_n^{2s+2}(t) &= \frac{1}{2^{s+1}}\frac{1}{\sqrt{1-t^2}}(1-T_{2n+2}(t))^{s+1}\\ &= \frac{1}{\sqrt{1-t^2}}\left(A_s + \sum_{j=1}^{s+1} B_{js} T_{(2n+2)j}(t)\right),\end{aligned} \tag{3.11}$$

where

$$A_s = \frac{1}{2^{2s+2}}\binom{2s+2}{s+1}, \qquad B_{js} = \frac{(-1)^j}{2^{2s+1}}\binom{2s+2}{s+1-j},$$

to obtain that

$$\begin{aligned}\int_{-1}^{1}(1-t^2)^{s+\frac{1}{2}} U_n^{2s+2}(t) T_n(t)Q(t)\mathrm{d}t &= A_s\int_{-1}^{1}\frac{1}{\sqrt{1-t^2}}T_n(t)Q(t)\mathrm{d}t + \sum_{j=1}^{s+1} B_{js}\int_{-1}^{1}\frac{1}{\sqrt{1-t^2}}T_{(2n+2)j}(t)T_n(t)Q(t)\mathrm{d}t\\ &= 0.\end{aligned}$$

The last equality is true since $T_nQ \in \pi_{2n-1}(\mathbb{R})$, and $T_{(2n+2)j}, j = 1, \ldots, s+1$, is orthogonal to all polynomials in $\pi_{2n-1}(\mathbb{R})$. This gives that the nodes of quadrature (3.10) are the zeros $\{\xi_j\}_{j=1}^n$ of T_n. Let us now consider the interpolation quadrature with these nodes. Its weights $\{a_j\}_{j=1}^n$ are

$$\begin{aligned} a_j &= \int_{-1}^{1}\sqrt{1-t^2}(1-t^2)^s U_n^{2s+2}(t)\frac{\Lambda_j(t;\boldsymbol{\xi})}{\Lambda_j(\xi_j;\boldsymbol{\xi})}\mathrm{d}t\\ &= A_s\int_{-1}^{1}\frac{1}{\sqrt{1-t^2}}\frac{\Lambda_j(t;\boldsymbol{\xi})}{\Lambda_j(\xi_j;\boldsymbol{\xi})}\mathrm{d}t\\ &= \frac{\pi}{2^{2s+2}n}\binom{2s+2}{s+1}, \quad j = 1, \ldots, n,\end{aligned}$$

where we have used relation (3.11), and the fact that

$$\int_{-1}^{1}\frac{1}{\sqrt{1-t^2}}\frac{\Lambda_j(t;\boldsymbol{\xi})}{\Lambda_j(\xi_j;\boldsymbol{\xi})}\mathrm{d}t = \frac{\pi}{n}.$$

This interpolation formula is actually Gaussian, since every $Q \in \pi_{2n-1}(\mathbb{R})$ can be written as $Q = T_nS + \tilde{Q}$, with $S, \tilde{Q} \in \pi_{n-1}(\mathbb{R})$, and therefore

$$\int_{-1}^{1}(1-t^2)^{s+\frac{1}{2}} U_n^{2s+2}(t)Q(t)\mathrm{d}t = \int_{-1}^{1}(1-t^2)^{s+\frac{1}{2}} U_n^{2s+2}(t)\tilde{Q}(t)\mathrm{d}t = \sum_{j=1}^{n} a_j\tilde{Q}(\xi_j) = \sum_{j=1}^{n} a_jQ(\xi_j).$$

The proof is completed. $\square$

Notice that using the same arguments as in Corollary 2.7, we arrive at the following statement for the Fourier–Tchebycheff coefficients.

Corollary 3.9. *There exists a quadrature formula*

$$\int_{-1}^{1}(1-t^2)^{s+1/2}T_n(t)f(t)\mathrm{d}t \approx \sum_{j=1}^{n}\sum_{i=0}^{2s+1} a_{ji}f^{(i)}(\eta_j)$$

with ADP $=(2s+3)n-1$. *The nodes* η_j *of this formula are the zeros of* U_n.

Corollary 3.10. *The quadrature*

$$\frac{2}{\pi}\int_{-1}^{1}\sqrt{1-t^2}U_n(t)f(t)\mathrm{d}t \approx \frac{1}{2^n n}\sum_{j=1}^{n} f[\eta_1,\ldots,\eta_n,\xi_j],$$

with $\{\eta_j\}_{j=1}^n$ *the zeros of* U_n *and* $\{\xi_j\}_{j=1}^n$ *the zeros of* T_n *is the only Gaussian quadrature (exact for all polynomials of degree* $3n-1$*), of the form* (3.7) *with* $\mu(t)=\sqrt{1-t^2}$ *and* $P_n=U_n$.

Proof. An application of Corollary 3.7 in the case of $k=n$, $P_n=2^{-n}U_n$, $[a,b]\equiv[-1,1]$, $\mu(t)=\sqrt{1-t^2}$ and Lemma 3.8 with $s=0$ completes the proof. □

Finally, note that an application of Corollary 3.7 in the case $k=n$, $P_n=2^{1-n}T_n$, and using the fact that T_n is the nth orthogonal polynomial on $[-1,1]$ with weight $\mu(t)=(1-t^2)^{-1/2}T_n^2(t)$ produces the formula

$$\int_{-1}^{1}\frac{1}{\sqrt{1-t^2}}T_n(t)f(t)\mathrm{d}t \approx 2^{n-1}\sum_{j=1}^{n} a_j f[\xi_1,\ldots,\xi_n,\xi_j],$$

where $\{\xi_j\}_{j=1}^n$ are the zeros of T_n and $\{a_j\}_{j=1}^n$ are the weights of the Gauss formula with n nodes on $[-1,1]$ with weight $\mu(t)=(1-t^2)^{-1/2}T_n^2(t)$. The formula is exact for all polynomials in $\pi_{3n-1}(\mathbb{R})$. This is exactly formula (1.2) derived in [1]. Indeed, subtracting (1.2) from the above formula, we conclude that the linear functional

$$2^{n-1}\sum_{j=1}^{n} a_j f[\xi_1,\ldots,\xi_n,\xi_j] - \frac{\pi}{n2^n}f'[\xi_1,\ldots,\xi_n]$$

annihilates all polynomials of degree $3n-1$. But it is based only on $2n$ evaluations: the values of f and f' at n points. Thus all coefficients of $f(\xi_k)$ and $f'(\xi_k)$, for $k=1,\ldots,n$, in this expression should be equal to zero. Hence, the formulas coincide. □

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