LINEAR TRANSPORT EQUATIONS WITH DISCONTINUOUS COEFFICIENTS

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1 Introduction.

Linear transport equations, while interesting in their own right, also occur in the study of nonlinear transport equations such as conservation laws. In this context, one encounters discontinuous transport velocities and the question of existence and uniqueness of solutions arises. Several recent papers (see [1, 6, 11]) have addressed these issues. However, there has not yet emerged a completely satisfactory existence-uniqueness theory. Indeed, it seems most natural to define the concept of a weak solution to such equations. In formulating the concept of weak solution, any initial condition which is simply continuous is allowed. In general, there will be infinitely many weak solutions. An (entropy) condition is needed to extract a unique weak solution. Certain entropy conditions were introduced by Bouchut and James (see [1]) but only in the restricted case that the initial condition is locally Lipschitz.

The purpose of the present paper is to introduce an entropy condition which picks out a unique weak solution for any continuous initial condition. This entropy condition will agree with that in [1] in the case the initial condition is Lipschitz. We shall study properties of the entropy solution including: (i) the relationship to conditions for extracting solutions (Filippov (see Conway [2]), reversible (see Bouchut and James [1])), (ii) minimality properties, (iii) properties of the solution operator.

We consider the one-dimensional homogeneous linear transport equation

$$u_t(t,x) + a(t,x)u_x = 0, \quad (t,x) \in \Omega := [0,T] \times \mathbb{R}, \tag{1.1}$$

with initial condition

$$u(0,x) = u^0(x), \tag{1.2}$$

where the velocity a is bounded and satisfies the one-sided Lipschitz condition

$$(a(t,x)-a(t,y))(x-y) \geq -m(t)(x-y)^2, \quad m \in L^1[0,T].$$

Here the coefficient a is possibly discontinuous and the theory of R. DiPerna and P. Lions [5] cannot be applied. The problem (1.1)-(1.2) has been studied by E. Conway [2], E. Tadmor [11], B. Perthame [6], F. Bouchut and F. James [1], Poupaud and Rascle [10], and others.

We are interested in the questions of existence, uniqueness, and regularity of solutions to (1.1)-(1.2). This problem has been considered before in [1], [2] and [11] and various theories have evolved with a different definition of a solution. For example (see [2]), Conway defines a solution to be any locally Lipschitz continuous function that solves the equation (1.1) almost everywhere and satisfies the initial condition (1.2). This is also the definition considered by Bouchut and James [1], and Tadmor [11].

In this setting, there is more than one Lipschitz solution to (1.1)-(1.2). In [1], Bouchut and James have introduced a criteria (see (2.8)) that picks out a unique solution to (1.1)-(1.2) whenever the initial data $u_0 \in Lip1$.

In the present paper we shall consider solutions to (1.1)-(1.2) in the weak sense (see (2.3)). In this case, to define a weak solution, one needs to assume that u^0 is only continuous. The existence of a solution is trivial (see Theorem 2.3) but the question of uniqueness remains. Our main result is to introduce an entropy condition which selects a unique solution from the set of continuous weak solutions to (1.1)-(1.2). This entropy solution has certain minimality properties (see §4 and §5) and can also be described by using the Filippov flow for the characteristic equation. When $u_0 \in Lip1$, our entropy condition is equivalent to the ones, introduced by Bouchut and James (see [1]) in the case of Lipschitz initial conditions.

In this paper we shall also discuss regularity theorems for the entropy solution operator. As a corollary of our results, we prove that for any continuous function u^0 of bounded variation there is a unique continuous solution to (1.1)-(1.2) that preserves the variation of the initial condition, i.e.

$$Var_{\mathbb{R}} u(t,\cdot) = Var_{\mathbb{R}} u^0, \quad \text{for all } t \in [0,T]. \tag{1.3}$$

2 Preliminaries.

We consider the Cauchy problem (1.1)-(1.2) with $a(t,\cdot)$ bounded for almost every t, i.e. for almost every t

$$\mid a(t,x) \mid \leq \parallel a \parallel_{L^\infty(\Omega)}, \text{ for all } x \in \mathbb{R}, \tag{2.1}$$

and satisfying the one-sided Lipschitz condition

$$(a(t,x)-a(t,y))(x-y) \geq -m(t)(x-y)^2, \text{ for all } x,y \in \mathbb{R}, \tag{2.2}$$

where $m \in L^1[0,\, T],\ m(t) \geq 0$ a.e. in $[0, T]$.

The problem (1.1) with initial condition (1.2) will be understood in weak sense. A function u is said to be a weak solution to (1.1)-(1.2) if

$$\int_0^T \int_{\mathbb{R}} u\Phi_t \, dxdt + \int_0^T [\int_{\mathbb{R}} u \, d(a\Phi)] \, dt + \int_{\mathbb{R}} \Phi(0,x) u^0(x) \, dx = 0 \tag{2.3}$$

for all test functions $\Phi \in C^\infty(\Omega)$ with compact support in $[0, T) \times \mathbb{R}$. The conditions (2.1)-(2.2) imply that for almost every $t \in (0, T)$, $a(t, \cdot) \in BV_{loc}(\mathbb{R})$ and (see [1])

$$Var_{[x,y]}(a(t,.)) \leq 2(\| a \|_{L^\infty(\Omega)} + m(t)(y - x)), \quad x < y. \tag{2.4}$$

Thus, the integrals in (2.3) are well defined (as Stieltjes integrals) whenever $u \in C(\Omega)$. Moreover, the continuity of u in the x direction is clearly a necessary condition to define weak solutions.

With (1.1) we associate the following characteristic ODE with possibly discontinuous right-hand side

$$\frac{d\chi}{ds} = a(s, \chi), \quad 0 \leq s \leq T, \tag{2.5}$$

$$\chi(t; (t, x)) = x. \tag{2.6}$$

If $a \in Lip(\Omega)$, the problem (2.5)-(2.6) has a unique classical solution $\chi \in C^1[0, T]$. In our case a is not smooth and therefore we need a generalization of the concept of a solution to this differential equation that includes the case in which the right-hand side of (2.5) is discontinuous. We choose the ideas of Filippov, which are used in the theory of conservation laws. This approach was also applied by Poupaud and Rascle (see [10]).

Filippov (see [7]) introduced the following definition of a solution to (2.5)-(2.6). We say that $\chi(s) := \chi(s; (t, x))$ is a Filippov solution to (2.5)-(2.6) on $[t_1, t_2]$, $t \in [t_1, t_2]$, if

(i) $\chi(s)$ is absolutely continuous on $[t_1, t_2]$,

(ii) $\chi(t) = x$,

(iii) for almost all $s \in (t_1, t_2)$

$$\lim_{\delta \to 0+} essinf_{|z - \chi(s)| < \delta} \{a(s, z)\} \leq \chi'(s) \leq \lim_{\delta \to 0+} esssup_{|z - \chi(s)| < \delta} \{a(s, z)\}.$$

Note that, if $a \in C(\Omega)$ the Filippov definition and the usual definition of a solution to (2.5)-(2.6) coincide, i.e. χ should have a continuous derivative $\chi'(s) = a(s, \chi)$ for all $s \in [t_1, t_2]$.

We will frequently use the trivial observation that if χ is a Filippov solution to (2.5)-(2.6) on the interval $[t_1, t_2]$, then

$$| \chi(s_1; (t, x)) - \chi(s_2; (t, x)) | \leq \| a \|_{L^\infty(\Omega)} | s_1 - s_2 | \tag{2.7}$$

for all $s_1,\ s_2 \in [t_1, t_2]$.

We shall first recall some results from [7], mentioned also in [10], that are proved under the assumptions (2.1)-(2.2) on a (Theorem 6, Theorem 10 in [7] and Theorem 2.2 in [10]).

THEOREM 2.1. (Filippov) *Let a satisfy the assumptions* (2.1)-(2.2)*. Then for every* $(t,x) \in [0,T] \times \mathbb{R}$ *there exist* $\underline{\chi}(s) := \underline{\chi}(s;(t,x))$ *and* $\overline{\chi}(s) := \overline{\chi}(s;(t,x))$*, called upper and lower Filippov solutions that pass through* (t,x) *with the following properties*

(a) $\quad \underline{\chi}(s) = \overline{\chi}(s), \quad 0 \le s \le t,$

(b) *for any Filippov solution χ of* (2.5)-(2.6) *we have*

$$\underline{\chi}(s) \le \chi(s) \le \overline{\chi}(s), \quad 0 \le s \le T,$$

(c) *for any $s \in [0,T]$ and any γ, where $\underline{\chi}(s) \le \gamma \le \overline{\chi}(s)$ there is a Filippov solution χ to* (2.5)-(2.6)*, such that $\chi(s) = \gamma$.*

THEOREM 2.2. (Filippov) *Let a satisfy the assumptions* (2.1)-(2.2)*. Then there is a constant $c = c(T)$ such that for every $t \in [0,T]$ and every Filippov solution χ*

$$\mid \chi(s;(t,x)) - \chi(s;(t,y)) \mid \le c(T) \mid \chi(t;(t,x)) - \chi(t;(t,y)) \mid = c(T) \mid x - y \mid,$$

for $0 \le s \le t,\ x, y \in \mathbb{R}$.

These theorems imply that if the coefficient a satisfies (2.1)-(2.2), there is a Filippov solution to problem (2.5)-(2.6) and it is uniquely determined on the left. Namely, starting from the point x at time t there is a unique characteristic going backward, defined for all $s \in [0,t]$.

REMARK 2.1. The intervals $(\underline{\chi}(s;(t,x)), \overline{\chi}(s;(t,x)))$ and $(\underline{\chi}(s;(t,y)), \overline{\chi}(s;(t,y)))$ do not intersect for all $x \neq y$ and all $s \in [t,T]$.

In order to prove this, let us assume that there exist time $s \in [t,T]$ and $\gamma \in \mathbb{R}$, such that

$$\gamma \in (\underline{\chi}(s;(t,x)), \overline{\chi}(s;(t,x))) \bigcap (\underline{\chi}(s;(t,y)), \overline{\chi}(s;(t,y))).$$

By Theorem 2.1, (c), we can find characteristics χ_1 and χ_2 such that

$$\gamma = \chi_1(s;(t,x)) = \chi_2(s;(t,y)).$$

Then χ_1 and χ_2 are Filippov characteristics passing through γ at time s and using the uniqueness on the left (Theorem 2.1, (a)), we obtain $\chi_1(\tau) = \chi_2(\tau)$ for any $\tau \le s$. When $\tau = t$, we get that $x = \chi_1(t) = \chi_2(t) = y$ which is a contradiction.

Bouchut and James (see [1]) study problem (1.1)-(1.2) with Lipschitz continuous initial data u^0. They introduce a notion of a *reversible* solution and *reversible* flow and show that this flow coincides with the Filippov solution χ to (2.5)-(2.6). Moreover, the *reversible* solution is unique and can be expressed explicitly in the form $u(t,x) := u^0(\chi(0;(t,x))$, where

$u^0 \in Lip1$ is the initial data (for details see Proposition 4.1.16 and Proposition 4.3.9 in [1]). Bouchut and James do not consider weak solutions, but in fact the function u, obtained above, is a weak solution to (1.1)-(1.2), and is the only one which is Lipschitz continuous and satisfies the equality

$$\int_{\mathbb{R}} |u_x(t,x)|\, dx = \int_{\mathbb{R}} |u_x^0(x)|\, dx \quad \text{for all } t \in [0,T]. \tag{2.8}$$

Note that, while (2.8) is proved only under the assumption $u^0 \in Lip1$, it has meaning for any u with $u_x(t,\cdot) \in L_1(\mathbb{R})$. More generally, for functions $u(t,\cdot) \in BV$ and $u^0 \in BV$, (1.3) is the analog of (2.8). Therefore it is natural to ask whether there is only one weak solution u to (1.1)-(1.2), with $u^0 \in C \bigcap BV$, that satisfies (1.3). We give a positive answer to this question (see Theorem 5.2).

Returning to the case when the initial condition u_0 is only continuous, let us observe that

$$u(t,x) := u^0(\chi(0;(t,x)) \tag{2.9}$$

is a weak solution to (1.1)-(1.2). For the proof of this fact it is enough to consider Lipschitz functions u_n^0 which converge to u^0 uniformly on any compact subset of Ω. Then, u is a weak solution to (1.1)-(1.2) because $u_n(t,x) := u_n^0(\chi(0;(t,x))$ is a weak solution to (1.1) with initial data u_n^0. In going further we shall call (2.9) the *Filippov solution* of (1.1)-(1.2). Hence we have the following existence result.

THEOREM 2.3. *If a satisfies the assumptions* (2.1)-(2.2) *and $u^0 \in C(\mathbb{R})$, then there exists a continuous weak solution to* (1.1)-(1.2). *Moreover, one such solution is the Filippov solution to* (1.1)-(1.2).

This brings us to the question of uniqueness of weak solutions in the case when $u_0 \in C(\mathbb{R})$. We introduce new entropy conditions (see Theorem 4.1, (i)) that do not require additional smoothness on the initial data, hence can be applied for any $u_0 \in C(\mathbb{R})$. We shall prove that the Filippov solution to (1.1)-(1.2) is the only weak solution to our problem that satisfies these entropy conditions. Therefore, using these conditions, we choose only one weak solution to (1.1)-(1.2) from the set of all weak solutions - namely the Filippov solution $u(t,x) := u^0(\chi(0;(t,x))$. In the special case when $u_0 \in Lip1$, this solution is the reversible solution introduced by Bouchut and James.

3 Regularization results and the method of characteristics.

Let us first begin by making some observations about the Filippov flow χ when a satisfies conditions (2.1)-(2.2). Consider any point $x \in \mathbb{R}$ and any interval I. Note that a single point can be considered as a closed interval. Denote by $I_t := \{x : \chi(0;(t,x)) \in I\}$ which is the image of I under the flow χ at time t. From Theorem 2.1 it follows that if $I \bigcap J = \emptyset$ then $I_t \bigcap J_t = \emptyset$. Also the image of an interval I is always an interval I_t. If I is open then I_t is open, and if I is closed then I_t is closed. The image of any point is either a point or a closed interval.

We define two types of points at the initial time $s = 0$. We say that x is of *type 1* if its image is a point at any time $s \in [0, T]$ and denote this set by Δ. Otherwise, we say it is a point of *type 2*. Observe that because of Theorem 2.1 x is of *type 1* if and only if

$$\underline{\chi}(T; (0, x)) = \overline{\chi}(T; (0, x)).$$

Also, when $x \neq y$ and both are of *type 2*, we use Remark 2.1 and obtain

$$[\underline{\chi}(s; (0, x)), \overline{\chi}(s; (0, x))] \bigcap [\underline{\chi}(s; (0, y)), \overline{\chi}(s; (0, y))] = \emptyset$$

for every $s \in [0, T]$. Therefore the set of points of *type 2* is countable, say $\{x_i\}_{i=1}^{\infty}$. Let us denote the <u>open</u> intervals

$$I_i(s) := (\underline{\chi}(s; (0, x_i)), \overline{\chi}(s; (0, x_i))),$$

and their disjoint union

$$\Lambda(s) := \bigcup_{i=1}^{\infty} I_i(s).$$

Note that, $I_i(s)$ may be the empty set for some s and i.

As a simple corollary of Theorem 2.1 and Theorem 2.2 (see [1] for a similar result), we have the following lemma.

Lemma 3.1. *For any time $s \in [0, T]$ and any interval I we have*

$$\frac{meas(I)}{c(T)} \leq meas(I_s) \leq meas(I) + 2s \mid\mid a \mid\mid_{L^\infty(\Omega)},$$

where $c(T)$ is the constant from Theorem 2.2.

Proof. The lower estimate follows from Theorem 2.2. The upper estimate follows from (2.7). □

Let us recall that Δ is the set of all points of *type 1* and denote its flow image at time s by

$$\Delta(s) := \{x \ : \ x = \chi(s; (0, y)), \ y \in \Delta\}.$$

Further, we shall use the following property of the flow.

Lemma 3.2. *For any $x \in \mathbb{R} \setminus \Lambda(\mathrm{s})$ and any $\epsilon > 0$ we have*

$$meas(\Delta(s) \cap (x - \epsilon, x + \epsilon)) > 0.$$

Proof. Fix $x \in \mathrm{I\!R} \setminus \Lambda(\mathrm{s})$ and $\epsilon > 0$. We have that

$$(x-\epsilon, x+\epsilon) \setminus \Lambda(s) = (x-\epsilon, x+\epsilon) \cap (\Delta(s) \cup A \cup B),$$

where $A := \{y : \exists y_0 \text{ of } \textit{type 2},\ y = \underline{\chi}(s;(0,y_0)) = \overline{\chi}(s;(0,y_0))\}$ and B is the set of all endpoints of the intervals in $\Lambda(s)$. Hence

$$meas(\Delta(s) \cap (x-\epsilon, x+\epsilon)) = meas((x-\epsilon, x+\epsilon) \setminus \Lambda(s))$$

because the sets A and B are at most countable. Since $x \notin \Lambda(s)$, the preimage $K := (\chi(0;(s,x-\epsilon)), \chi(0;(s,x+\epsilon)))$ of $(x-\epsilon, x+\epsilon)$ is a non-trivial interval.

Suppose that $meas((x-\epsilon, x+\epsilon) \cap \Delta(s)) = 0$. Then $meas((x-\epsilon, x+\epsilon) \cap \Lambda(s)) = 2\epsilon$. Recall that $\Lambda(s) = \bigcup_{i=1}^{\infty} I_i(s)$. So, for any $0 < \lambda < 2$ there exists an index set $\mathcal{I}_\lambda \subseteq \{1, 2, \ldots\}$, $\#\mathcal{I}_\lambda < \infty$, such that

$$meas((x-\epsilon, x+\epsilon) \cap \bigcup_{i \in \mathcal{I}_\lambda} I_i(s)) \geq (2-\lambda)\epsilon.$$

We can write $(x-\epsilon, x+\epsilon) \setminus \bigcup_{i \in \mathcal{I}_\lambda} I_i(s) = \bigcup_{j \in \mathcal{J}_\lambda} J_j(s)$, where $\#\mathcal{J}_\lambda < \infty$ and $J_j(s)$ are disjoint intervals. Let $J_j(0)$ be the preimage of $J_j(s)$. From Lemma 3.1, applied to each $J_j(0)$, we have

$$meas(K) = meas(\bigcup_{j \in \mathcal{J}_\lambda} J_j(0)) \leq c(s) meas(\bigcup_{j \in \mathcal{J}_\lambda} J_j(s)) \leq c(s)\lambda\epsilon.$$

Here the equality uses the fact that the points of *type 2* are countable. Taking $\lambda < \frac{meas(K)}{2c(s)\epsilon}$, we get a contradiction. □

Note that, by Theorem 2.1 and the properties of the Filippov flow (see [1] for a similar result), we obtain that, for each $i = 1, 2, \ldots$, the Filippov solution $u(t,x) := u^0(\chi(0;(t,x))$ of (1.1)-(1.2) is constant on the set $\bigcup_{s \in [0,T]}(s, I_i(s))$, i.e.

$$u(t,x) = u^0(x_i) \text{ for all } (t,x) \in \bigcup_{s \in [0,T]} (s, I_i(s)). \tag{3.1}$$

The main result of this section is the following theorem.

THEOREM 3.1. *If v_1 and v_2 are two continuous weak solutions of* (1.1)-(1.2), *where the transport velocity a satisfies assumptions* (2.1)-(2.2), *then*

$$v_1(t,x) = v_2(t,x), \quad x \notin \Lambda(t), \quad 0 \leq t \leq T.$$

The remainder of this section is devoted to the proof of this theorem. Many aspects of the proof are standard. When this is the case, we shall defer this portion of the proof to the appendix. The idea of the proof is to approximate weak solutions to (1.1)-(1.2) by smooth solutions to the smooth backward problem. We introduce the following smoothing operators. Let ρ be a C^∞ non-negative function, supported on $[-1,1]$ and denote by $\rho^\delta(x) := \frac{1}{\delta}\rho(\frac{x}{\delta})$. Let η be a C^∞ non-negative function, supported on $[-1,0]$ and denote by $\eta^\epsilon(x) := \frac{1}{\epsilon}\eta(\frac{x}{\epsilon})$. We use ρ for smoothing in the x direction and η for smoothing in the t direction.

Given $g \in L^1_{loc}(\Omega)$, we define

$$S_{\epsilon,\delta}g(t,x) := g * \eta^\epsilon(t) * \rho^\delta(x).$$

$S_{\epsilon,\delta}g$ is well defined for $(t,x) \in [0, T-\epsilon] \times \mathbb{R}$. When $\epsilon = 0$, $S_{0,\delta}g := g * \rho^\delta$.

Fix a finite interval $I := [\alpha, \beta]$ and $\Omega_0 := [0,T] \times [\alpha, \beta]$. In this setting the following lemma is true.

Lemma 3.3. *Let a satisfy assumptions* (2.1)-(2.2). *Then there is a constant $M > 0$, independent of δ, such that*

$$\|a - S_{0,\delta}a\|_{L_1(\Omega_0)} \leq M\delta \tag{3.2}$$

for any $\delta \leq T \mid\mid a \mid\mid_{L^\infty(\Omega)}$.

Proof. Let $\alpha_1 := \alpha - 10T\|a\|_{L^\infty(\Omega)}$ and $\beta_1 := \beta + 10T\|a\|_{L^\infty(\Omega)}$. Because of (2.4), if we follow a standard approximation approach (see [4], page 53, Lemma 9.2), we get that

$$\int\int_{\Omega_0} \mid a(t,x) - a^\delta(t,x) \mid \, dxdt \leq c\delta \int_0^T Var_{[\alpha_1,\beta_1]} a(t,\cdot)\, dt$$

$$\leq 2c\delta(T \mid\mid a \mid\mid_{L^\infty(\Omega)} + (\beta_1 - \alpha_1)\int_0^T m(t)\, dt) = M\delta,$$

where M is an absolute constant independent of δ. □

Let us introduce some notation. For a function $v \in C(\Omega)$, $v(0,x) = 0$, we denote by $v^\epsilon := S_{\epsilon,\epsilon}v$ and $r^\epsilon := v^\epsilon_t + av^\epsilon_x$. Note that, v^ϵ is well defined for a fixed compact set $K \subset [0,T) \times \mathbb{R}$ provided ϵ is small. Then, similar to DiPerna and Lions (see [5], Theorem II.1), we can prove the following theorem.

Theorem 3.2. *Let $v \in C(\Omega)$ be a continuous weak solution to* (1.1) *with zero initial data $v(0,x) = 0$. Then for any compact set $K \subset [0,T) \times \mathbb{R}$*

$$\|r^\epsilon\|_{L^1(K)} \longrightarrow 0 \quad \textit{as } \epsilon \textit{ goes to } 0.$$

We give a complete proof of this statement in the appendix.

Remark 3.1. We could equally as well prove the same result for $r^{\epsilon,\delta} := (S_{\epsilon,\delta}v)_t + a(S_{\epsilon,\delta}v)_x$ with any $\epsilon, \delta \longrightarrow 0$.

Fix s, $0 < s < T$. Let $a^\epsilon := S_{0,\epsilon^2}a$ and consider the smooth, with respect to x, backward problem

$$w_t + a^\epsilon \; w_x = 0, \quad (t,x) \in [0,s] \times \mathbb{R}, \tag{3.3}$$

$$w(s,x) = v^\epsilon(s,x), \quad x \in \mathbb{R}. \tag{3.4}$$

For any $\epsilon > 0$, the problem (3.3)-(3.4) has a unique continuous weak solution w^ϵ (see [1], [5] and [6]), given by the formula (see [1])

$$w^\epsilon(t,x) = v^\epsilon(s, \chi^\epsilon(s;(t,x))),$$

where χ^ϵ is the Filippov solution to

$$\frac{d\chi}{ds} = a^\epsilon(s,\chi), \quad 0 \le s \le T,$$

$$\chi(t;(t,x)) = x.$$

If we denote $e^\epsilon := v^\epsilon - w^\epsilon$, $\phi^\epsilon := r^\epsilon + (a^\epsilon - a)v^\epsilon_x$, then e^ϵ will be the continuous solution to the problem

$$e^\epsilon_t + a^\epsilon e^\epsilon_x = \phi^\epsilon, \quad (t,x) \in (0,s) \times \mathbb{R}, \tag{3.5}$$

$$e^\epsilon(s,x) = 0 \quad x \in \mathbb{R}. \tag{3.6}$$

From the fact that $\mid v^\epsilon_x \mid \le \frac{c(\Omega_0)}{\epsilon}$ and $v(0,x) = 0$, using (3.2) and Theorem 3.2, we obtain $\phi^\epsilon \longrightarrow 0$ in $L^1_{loc}([0,T) \times \mathbb{R})$ as $\epsilon \to 0$. Similarly to [1], we have the following lemma.

Lemma 3.4. *Let a satisfy* (2.1)-(2.2) *and e^ϵ be the continuous solution to* (3.5)-(3.6) *with $\phi^\epsilon \longrightarrow 0$ in $L^1_{loc}([0,s] \times \mathbb{R})$. Then $e^\epsilon(0,\cdot) \longrightarrow 0$ in $L^1_{loc}(\mathbb{R})$ as $\epsilon \longrightarrow 0$.*

Proof. This can be proved by standard integration on a cone, similarly to Lemma 4.1.1 in [1], using the fact that $\|\phi^\epsilon\|_{L^1_{loc}([0,T)\times\mathbb{R})} \longrightarrow 0$ as $\epsilon \to 0$ (see the appendix). □

Recall that the function v^ϵ is defined on $[0, T-\epsilon] \times \mathbb{R}$, and $v \in C([0,T] \times \mathbb{R})$ is a weak solution to (1.1)-(1.2) with zero initial data. Therefore, given a compact set $K \subset [0,T) \times \mathbb{R}$, we have that v^ϵ is well defined on K for ϵ small enough. Moreover, v^ϵ converges to v uniformly on K. In particular, $v^\epsilon(0,\cdot)$ tends uniformly to zero on each compact subset of $\mathbb{R}$. As a result of Lemma 3.4, we get

$$w^\epsilon(0,\cdot) \longrightarrow 0$$

almost everywhere on $\mathbb{R}$. On the other hand, the solution of (3.3)-(3.4) is

$$w^\epsilon(t,x) = v^\epsilon(s, \chi^\epsilon(s;(t,x))).$$

If x is of *type 1*, then (see [7], Theorem 11)

$$\chi^\epsilon(s;(0,x)) \longrightarrow \chi(s;(0,x)),$$

the unique value along the characteristic through $(0,x)$ and therefore

$$v^\epsilon(s, \chi^\epsilon(s;(0,x))) \longrightarrow v(s, \chi(s;(0,x))). \tag{3.7}$$

Fix a closed finite interval I. Then (3.7) implies that

$$v(s; \chi(s;(0,x))) = 0, \tag{3.8}$$

for almost all points x of *type 1* in I, $0 < s < T$.

Now, our goal is to use (3.8) and prove that $v(s,x) = 0$ holds for <u>all</u> points $x \notin \Lambda(s)$. We will use the continuity of v and Lemma 3.2. Since I is a closed interval, I_s is also closed. Therefore $v(s,\cdot)$ is uniformly continuous on I_s. Fix $\epsilon > 0$. Then $\exists \delta$, such that $\mid v(s,x_1) - v(s,x_2) \mid < \epsilon$, whenever $\mid x_1 - x_2 \mid < \delta$, $x_1, x_2 \in I_s$. Fix $x \in int(I_s)$, $x \notin \Lambda(s)$. By Lemma 3.2, we have $meas(\Delta(s) \bigcap (x-\delta, x+\delta)) > 0$. Therefore, using (3.8) there is $y \in \Delta(s) \bigcap (x-\delta, x+\delta)$ such that $v(s,y) = 0$. Since $\mid y - x \mid < \delta$, it follows that $\mid v(s,x) \mid < \epsilon$. But ϵ was arbitrary, hence

$$v(s,x) = 0 \text{ for any } x \notin \Lambda(s) \text{ and } s \in [0,T). \tag{3.9}$$

For $s = T$, we get the same result because we assume that v is continuous on $[0,T] \times \mathbb{R}$.

In order to complete the proof of Theorem 3.1, we take two continuous weak solutions v_1, v_2 to (1.1)-(1.2). Then, the difference $v := v_1 - v_2$ is a continuous weak solution to (1.1)-(1.2) with zero initial data and we finish the proof using (3.9).

4 An entropy condition for the Filippov solution of the linear transport equation.

Let $I := [\alpha, \beta]$ be a finite interval, $u^0 \in C(I)$, and $u(t,x) = u^0(\chi(0;(t,x)))$ be the Filippov solution of (1.1)-(1.2). By a partition of I, we mean a collection $\Gamma := \{J\}$, where J are disjoint intervals, whose union is I. Let us fix $n \geq 0$ and define the nonlinear space of piecewise constants on I with at most $(n+1)$ pieces by

$$\Sigma_n(I) := \{g : g(x) = \sum_{J \in \Gamma} a_J \varphi_J(x),\ a_J \in R,\ \#\Gamma \leq n+1\},$$

where φ_J is the characteristic function of J. Given an interval I, we define the error of approximation of a function $f \in C(I)$ by elements from $\Sigma_n(I)$ by

$$\sigma_n(f)_I := \sigma_n(f)_{C(I)} = inf_{g\in\Sigma_n(I)} \mid\mid f - g \mid\mid_{C(I)} .$$

Recall that the flow image at time t of any interval I is the interval $I_t = \{x : \chi(0; (t, x)) \in I\}$.

Theorem 4.1. *Let u be the Filippov and v be any weak continuous solution of* (1.1)-(1.2)*, where the transport velocity a satisfies assumptions* (2.1)-(2.2)*. Let I be an arbitrary finite interval and I_t be its flow image, $0 < t \leq T$. Then*

$$\text{(i)} \qquad \sigma_n(u^0)_I = \sigma_n(u(t, \cdot))_{I_t}, \quad n = 0, 1, 2, \dots ,$$

$$\text{(ii)} \qquad \sigma_n(u(t, \cdot))_{I_t} \leq \sigma_n(v(t, \cdot))_{I_t}, \quad n = 0, 1, 2, \dots .$$

If there is (t_0, x_0) for which $u(t_0, x_0) \neq v(t_0, x_0)$, then there exists an index $n_0 \geq 0$, such that for any finite interval I^* with $[x_0 - t_0\|a\|_{L^\infty(\Omega)}, x_0 + t_0\|a\|_{L^\infty(\Omega)}] \subset I^*$, we have

$$\text{(iii)} \qquad \sigma_{n_0}(u(t_0, \cdot))_{I^*_{t_0}} < \sigma_{n_0}(v(t_0, \cdot))_{I^*_{t_0}} .$$

Proof. Let $g \in \Sigma_n(I)$, $g(x) = \sum_{J\in\Gamma} a_J\varphi_J(x)$. For any $J \in \Gamma$, we consider $J_t := \{x : \chi(0; (t, x)) \in J\}$. Then $\Gamma_t := \{J_t\}_{J\in\Gamma}$ is a partition of I_t and

$$g^t(x) := \sum_{J_t\in\Gamma_t} a_J\varphi_{J_t}(x)$$

is an element from $\Sigma_n(I_t)$. Therefore, every $g \in \Sigma_n(I)$ will generate an element from $\Sigma_n(I_t)$, namely g^t.

For any $u^0 \in C(I)$ (see [4], Theorem 4.1, page 363) we can find an element of best approximation from $\Sigma_n(I)$. More precisely, there is

$$g_0(x) = \sum_{J\in\Gamma} a^0_J\varphi_J(x) \in \Sigma_n(I),$$

such that

$$\sigma_n(u^0)_I =\mid\mid u^0 - g_0 \mid\mid_{C(I)} .$$

But $g^t_0 \in \Sigma_n(I_t)$ and therefore

$$\mid\mid u(t, \cdot) - g^t_0(\cdot) \mid\mid_{C(I_t)} \geq \sigma_n(u(t, \cdot))_{I_t}.$$

Also, since u is the Filippov solution of (1.1)-(1.2), for every $x \in I_t$ there is $y \in I$, $y = \chi(0; (t, x))$, with the property $u^0(y) - g_0(y) = u(t, x) - g^t_0(x)$ and vice versa. Therefore, we have

$$\sigma_n(u^0)_I =\mid\mid u^0 - g_0 \mid\mid_{C(I)} =\mid\mid u(t, \cdot) - g^t_0(\cdot) \mid\mid_{C(I_t)} \geq \sigma_n(u(t, \cdot))_{I_t}. \tag{4.1}$$

We can reverse the inequality (4.1) as follows. Any interval $J_t \subset I_t$ has a flow preimage $J \subset I$. Hence, given a partition $\Gamma_t = \{J_t\}$ of I_t with $n+1$ intervals, we can generate a partition Γ of I consisting of the preimages J of J_t, $J_t \in \Gamma_t$. We omit these J that are $\emptyset$. The same way as above, we obtain that

$$\sigma_n(u(t,\cdot))_{I_t} = \| u(t,\cdot) - h_0^t(\cdot) \|_{C(I_t)} = \| u^0 - h_0 \|_{C(I)} \geq \sigma_n(u^0)_I. \tag{4.2}$$

Here

$$h_0^t(x) := \sum_{J_t \in \Gamma_t} c_{J_t} \varphi_{J_t}(x) \in \Sigma_n(I_t),$$

is the element of best approximation to $u(t,\cdot)$ from $\Sigma_n(I_t)$ and

$$h_0(x) := \sum_{J \in \Gamma} c_J \varphi_J(x) \in \Sigma_n(I).$$

Hence, (4.1) and (4.2) imply that for the Filippov solution u of (1.1)-(1.2) we have

$$\sigma_n(u(t,\cdot))_{I_t} = \sigma_n(u^0)_I. \tag{4.3}$$

Let v be any other continuous weak solution of (1.1)-(1.2) and

$$f^t(x) := \sum_{J_t \in \Gamma_t} b_{J_t} \varphi_{J_t}(x) \in \Sigma_n(I_t),$$

be the element of best approximation to $v(t,.)$ from $\Sigma_n(I_t)$, namely

$$\sigma_n(v(t,\cdot))_{I_t} = \| v(t,\cdot) - f^t(\cdot) \|_{C(I_t)} .$$

Observe that if we fix an element of best approximation f^t, we fix a partition Γ_t ($\#\Gamma_t \leq n+1$) of I_t.

We will now modify f^t but only on the set $\Lambda(t)$ as follows. Let $x_i \in I$, with x_i a point of *type 2*. Consider the interval $I_i(t)$. We know that $I_i(t) \in I_t$. We order the partition Γ_t using the natural order on the real line. Then there exist at most two intervals of the partition Γ_t, we denote them by J_t^l and J_t^r, such that J_t^l is the first with nonempty intersection with $I_i(t)$, and J_t^r is the last with nonempty intersection with $I_i(t)$. We use this notation because if J_t^l and J_t^r do not coincide, then J_t^l is to the left of J_t^r.

Now, we can define a new function $\overline{f}^t \in \Sigma_n(I_t)$ as follows

$$\overline{f}^t(x) := \begin{cases} f^t(x) & \text{if } x \in I_t \setminus \Lambda(t), \\ b_{J_t^l} \varphi_{(J_t^l \bigcap I_i(t))}(x) + b_{J_t^r} \varphi_{(I_i(t) \setminus J_t^l)}(x) & \text{if } x \in I_i(t). \end{cases} \tag{4.4}$$

Note that $\overline{f}^t \in \Sigma_n(I_t)$, since we can only decrease the number of intervals of the partition Γ_t. Therefore

$$\| u(t,\cdot) - \overline{f}^t(\cdot) \|_{C(I_t)} \geq \sigma_n(u(t,\cdot))_{I_t}. \tag{4.5}$$

The statement (3.1) implies that u is constant on $I_i(t),\ i=1,2,\ldots$, and using this, we can easily obtain that

$$\text{(4.6)} \qquad || \ u(t,\cdot)-f^t(\cdot)\ ||_{C(I_t\setminus\Lambda(t))}=||\ u(t,\cdot)-\overline{f}^t(\cdot)\ ||_{C(I_t)}\ .$$

Also, since $v(t,x)=u(t,x),\ x\in I_t\setminus\Lambda(t)$ (see Theorem 3.1), we have

$$\text{(4.7)} \qquad ||\ v(t,\cdot)-f^t(\cdot)\ ||_{C(I_t\setminus\Lambda(t))}=||\ u(t,\cdot)-f^t(\cdot)\ ||_{C(I_t\setminus\Lambda(t))}\ .$$

When we combine (4.5), (4.6), (4.7), we derive

$$\text{(4.8)} \qquad \sigma_n(v(t,\cdot))_{I_t}=||\ v(t,\cdot)-f^t(\cdot)\ ||_{C(I_t)}\geq||\ v(t,\cdot)-f^t(\cdot)\ ||_{C(I_t\setminus\Lambda(t))}\geq\sigma_n(u(t,\cdot))_{I_t}.$$

This proves (ii).

Let $u(t,x)\neq v(t,x)$. Then there is (t_0,x_0) with the property

$$\text{(4.9)} \qquad |\ u(t_0,x_0)-v(t_0,x_0)\ |=10\delta,$$

for some $\delta>0$. Also, $x_0\in\Lambda(t_0)$ and let $I_{i_0}(t_0):=(\overline{\alpha},\overline{\beta})$ be the component of $\Lambda(t_0)$ that contains x_0. Then since $u(t,x)$ is a constant on $I_{i_0}(t_0)$ (see (3.1)) and by Theorem 3.1, we have

$$\text{(4.10)} \qquad v(t_0,\overline{\alpha})=u(t_0,\overline{\alpha})=u(t_0,x)=u(t_0,\overline{\beta})=v(t_0,\overline{\beta}),\quad x\in I_{i_0}(t_0).$$

Moreover, using (4.9) and (4.10), we derive

$$\text{(4.11)} \qquad Var_{(\overline{\alpha},\overline{\beta})}v\geq 20\delta>Var_{(\overline{\alpha},\overline{\beta})}u=0.$$

We know that $I_{i_0}(t_0)$ is generated by a point x_{i_0} of *type 2*. Using the properties of the Filippov flow (see [1]), we obtain that $x_{i_0}\in[x_0-t_0\|a\|_{L^\infty(\Omega)},x_0+t_0\|a\|_{L^\infty(\Omega)}]$. Then for any interval I^* with $[x_0-t_0\|a\|_{L_\infty(\Omega)},x_0+t_0\|a\|_{L_\infty(\Omega)}]\subset I^*$, we have that $x_{i_0}\in I^*$ and therefore $I_{i_0}(t_0)\subset I^*_{t_0}$. Moreover, the set $I^*_{t_0}\setminus\Lambda(t_0)$ is non-empty because the points of *type 2* are countably many, so their union can not be the whole interval I^*.

If $u(t_0,x)=const$ for $x\in I^*_{t_0}$, then

$$\sigma_0(u(t_0,\cdot))_{I^*_{t_0}}=0<\delta<\sigma_0(v(t_0,\cdot))_{I^*_{t_0}},$$

and (iii) holds for $n_0=0$.

If $u(t_0,x)\neq const$ for $x\in I^*_{t_0}$, then $u^0(x)\neq const$ for $x\in I^*$. Hence $\sigma_n(u_0)_{I^*}>0$, $n=0,1,\ldots$ and $\{\sigma_n(u^0)_{I^*}\}_{n=0}^\infty$ will be a non-increasing sequence of positive numbers that converges to 0. Therefore $\exists n_0=n_0(\delta)\geq 1$ with the property

$$\text{(4.12)} \qquad \sigma_{n_0}(u^0)_{I^*}<\sigma_{n_0-1}(u^0)_{I^*}\leq\delta.$$

Let us assume that

$$\sigma_n(u(t_0,\cdot))_{I^*_{t_0}} = \sigma_n(v(t_0,\cdot))_{I^*_{t_0}} \text{ for all } n.$$

Then (4.3) and (4.12) imply

$$\sigma_{n_0}(v(t_0,\cdot))_{I^*_{t_0}} = \sigma_{n_0}(u(t_0,\cdot))_{I^*_{t_0}} < \sigma_{n_0-1}(u(t_0,\cdot))_{I^*_{t_0}} = \sigma_{n_0-1}(v(t_0,\cdot))_{I^*_{t_0}}, \tag{4.13}$$

$$\sigma_{n_0}(v(t_0,\cdot))_{I^*_{t_0}} < \delta. \tag{4.14}$$

From (4.9), (4.10) and (4.14), it follows that if $f^{t_0} \in \Sigma_{n_0}(I^*_{t_0})$ is the best approximation to $v(t_0,\cdot)$, at least 3 intervals of the partition Γ_{t_0} (generated by f^{t_0}) will have nonempty intersection with $I_{i_0}(t_0)$. Therefore, if we consider again $\overline{f}^{t_0}$, given in (4.4), we will get that $\overline{f}^{t_0} \in \Sigma_{n_0-1}(I^*_{t_0})$ and likewise (see (4.6) and (4.7)) we will obtain the estimates

$$\sigma_{n_0}(v(t_0,\cdot))_{I^*_{t_0}} = || \, v(t_0,\cdot) - f^{t_0}(\cdot) \, ||_{C(I^*_{t_0})} \geq || \, v(t_0,\cdot) - f^{t_0}(\cdot) \, ||_{C(I^*_{t_0}\setminus\Lambda(t_0))}$$

$$= || \, u(t_0,\cdot) - f^{t_0}(\cdot) \, ||_{C(I^*_{t_0}\setminus\Lambda(t_0))} = || \, u(t_0,\cdot) - \overline{f}^{t_0}(\cdot) \, ||_{C(I^*_{t_0})} \geq \sigma_{n_0-1}(u(t_0,\cdot))_{I^*_{t_0}}.$$

But this inequality contradicts (4.13) and this proves part (iii) of Theorem 4.1 □

REMARK 4.1. Condition (i) can be viewed as an entropy condition that picks out only one solution among all continuous weak solutions of (1.1)-(1.2), namely the Filippov solution.

In the case of a compactly supported initial condition u^0, Theorem 4.1 can be formulated in the following way.

THEOREM 4.2. *Let u be the Filippov and v be any weak continuous solution of* (1.1)-(1.2)*, where the transport velocity a satisfies assumptions* (2.1)-(2.2)*. Then*

$$\text{(i)} \qquad \sigma_n(u^0)_{\mathbb{R}} = \sigma_n(u(t,\cdot))_{\mathbb{R}}, \quad n = 0,1,2,\dots,$$

$$\text{(ii)} \qquad \sigma_n(u(t,\cdot))_{\mathbb{R}} \leq \sigma_n(v(t,\cdot))_{\mathbb{R}}, \quad n = 0,1,2,\dots.$$

If there is (t_0,x_0), such that $u(t_0,x_0) \neq v(t_0,x_0)$, then there is an index $n_0 \geq 0$ for which

$$\textit{(iii)} \qquad \sigma_{n_0}(u(t_0,\cdot))_{\mathbb{R}} < \sigma_{n_0}(v(t_0,\cdot))_{\mathbb{R}}.$$

5 Minimal properties of the Filippov solution and regularity theorems for the solution operator.

In this section we discuss properties of the Filippov (entropy) solution to problem (1.1)-(1.2) with compactly supported initial data u^0 and prove certain regularity results for the solution operator. Let us define $C_0 := \{f \,:\, f \in C(\mathbb{R}) \text{ and } \exists\, L > 0 \text{ such that } supp(f) \subset [-L, L]\}$. In this setting, we have the following theorem.

Theorem 5.1. *Let u be the Filippov solution and v be any other continuous weak solution to* (1.1)-(1.2) *with $u_0 \in C_0 \bigcap BV$, where the transport velocity a satisfies assumptions* (2.1)-(2.2). *Then for any $t \in [0, T]$, we have*

$$Var_{\mathbb{R}}(u^0) = Var_{\mathbb{R}}(u(t,\cdot)) \leq Var_{\mathbb{R}}(v(t,\cdot)).$$

Moreover, there is time $t_0 \in (0, T]$, such that

$$Var_{\mathbb{R}}(u(t_0,\cdot)) < Var_{\mathbb{R}}(v(t_0,\cdot)).$$

Proof . From Theorem 4.2 and Kahane's theorem (see [4], page 365, Theorem 4.3) we conclude

$$Var_{\mathbb{R}}(u^0) = Var_{\mathbb{R}}(u(t,\cdot)) \tag{5.1}$$

for any $t \in [0, T]$ (this could also be proved directly). Let v be a continuous weak solution to (1.1)-(1.2) different from u. Theorem 4.1, part (ii), and Kahane's theorem give

$$Var_{I_t}(u(t,\cdot)) \leq Var_{I_t}(v(t,\cdot)) \tag{5.2}$$

for any interval $I \subset \mathbb{R}$ and any $t \in [0, T]$. Moreover, there is $(t_0, x_0) \in (0, T] \times \mathbb{R}$ and an interval $(\overline{\alpha}, \overline{\beta})$ (see (4.9-11)), such that

$$Var_R(v(t_0,\cdot)) \geq Var_{(-\infty,\overline{\alpha})}(v(t_0,\cdot)) + 20\delta + Var_{(\overline{\beta},\infty)}(v(t_0,\cdot)).$$

Using the definition of $\overline{\alpha}$ and $\overline{\beta}$, we conclude that there exists $x_0^* \in \mathbb{R}$ such that $(-\infty, x_0^*)_{t_0} = (-\infty, \overline{\alpha})$ and $(x_0^*, \infty)_{t_0} = (\overline{\beta}, \infty)$. Then from (5.2), applied for the intervals $(-\infty, x_0^*)$ and (x_0^*, ∞), and (4.10), it follows

$$Var_{\mathbb{R}}(v(t_0,\cdot)) \geq Var_{\mathbb{R}}(u(t_0,\cdot)) + 20\delta$$

for some $\delta > 0$. $\square$

Remark 5.1. In the case $u_0 \in C_0 \bigcap BV$, Theorem 5.1 provides us with another way of selecting the Filippov solution from the set of all continuous weak solutions. Namely, the

Filippov solution is the only one that preserves the variation of the initial condition for all $t \in [0, T]$.

Let us define the solution operator $U(t)$ by

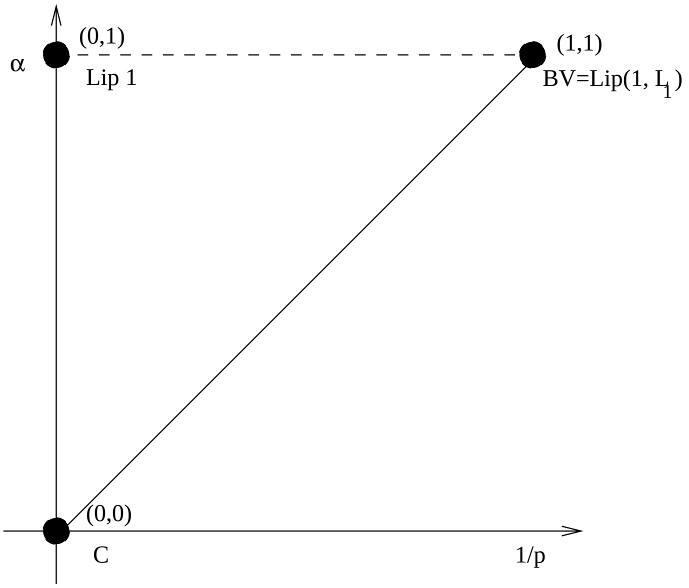
$$U(t)(u^0) := u(t, \cdot), \tag{5.3}$$

where u is the Filippov solution with initial condition u_0. We turn our attention to the the following questions:

(i) Which spaces are left invariant under $U(t)$, $t > 0$? That is, for which spaces do we have $U(t) : \mathrm{X} \to \mathrm{X}$ boundedly?

(ii) For which spaces X do we have $\|U(t)\|_{\mathrm{X}\to\mathrm{X}} \leq 1$?

It will be useful to have a way of visualizing spaces of functions as they occur in our discussion. We shall do this by using points in the upper right quadrant of the plane. The $x-$axis will correspond to the L_p spaces except that L_p is identified with $x = 1/p$ not with $x = p$. The $y-$axis will correspond to the order of smoothness. For example $y = 1$ will mean a space of smoothness order 1 (or one time differentiable if you like). Thus $(1/p, \alpha)$ corresponds to a space of smoothness α measured in the L_p-norm. For example we could identify this point with the space $Lip(\alpha, L_p)$ although we may want to vary this interpretation slightly. The following figure represents the spaces we shall discuss further.



Recall that the Filippov solution to (1.1)-(1.2) is given by $u(t, x) = u^0(\chi(0;\ (t, x)))$. From this representation it automatically follows that the solution operator $U(t)$ is a norm one operator from C_0 to C_0. Using Theorem 5.1, we have that $U(t)$ is a norm one operator from $C_0 \cap BV$ to $C_0 \cap BV$. The real interpolation spaces are defined by means of the K-functional. That is, given a pair of Banach spaces (X, Y), with Y continuously embedded in X: $Y \subset X$, for any $f \in X$ we define

$$K(f, \tau) := K(f, \tau; X, Y) := inf_{g \in Y} \{\|f - g\|_X + \tau \|g\|_Y\}, \quad \tau \geq 0.$$

For more details and properties of the K-functional and the real interpolation spaces see Chapter 6 of the book of DeVore and Lorentz [4]. The norm in the real interpolation space $(X, Y)_{\theta,q}$ is defined by

$$\|f\|_{(X,Y)_{\theta,q}} := \|f\|_X + \|K(f, \cdot)\|_{\theta,q},$$

where $\| \cdot \|_{\theta,q}$ is the θ, q-quasinorm of §3, Chapter 2, [4]. We have the following theorems.

Theorem 5.2. *$U(t)$ is a norm one operator on the real interpolation spaces $(C_0, C_0 \cap BV)_{\theta,q}$ for any $t \in [0, T]$.*

Proof . We know that $U(t)$ is a linear operator which is norm one from C_0 to C_0 and from $C_0 \cap BV$ to $C_0 \cap BV$. Therefore, by Theorem 7.1, Chapter 6, [4], $U(t)$ will map every real interpolation space $(C_0, C_0 \cap BV)_{\theta,q}$ into itself with norm one for all $0 < q \leq \infty$ and $0 < \theta < 1$. □

Remark 5.2. The real interpolation spaces for the pair $(C_0, C_0 \cap BV)$ can be defined via nonlinear approximation and in some cases Besov spaces. We will not formulate results in this context but refer the reader to [3, 9, 8].

Theorem 5.3. *$U(t)$ is bounded on any real interpolation space in the triangle with vertices C_0, $Lip1$, $C_0 \bigcap BV$ for any $t \in [0, T]$. That is, given a couple spaces (X, Y) such that X and Y are real interpolation spaces for any of the pairs $(C_0, C_0 \cap BV)$, $(C_0, Lip1)$ or $(C_0 \cap BV, Lip1)$, we have that $U(t)$ is bounded on the real interpolation space $(X, Y)_{\theta,q}$, for any $0 < \theta < 1$ and $0 < q \leq \infty$.*

Proof . From Theorem 5.2, it follows that the solution operator $U(t)$ is norm one on C_0 and $C_0 \cap BV$ for all $t \in [0, T]$. It is known (see [1, 11]) that $U(t)$ is bounded on $Lip1$. Using all of the above, by standard interpolation arguments, we obtain that $U(t)$ boundedly maps any interpolation space in the triangle with vertices C_0, $Lip1$, $C_0 \bigcap BV$ into itself. □

Corollary. $U(t)$ is bounded on $B^\alpha_q(L_p)$ for $1 \leq p \leq \infty$, $q \geq 0$ and $1/p < \alpha < 1$.

6 Appendix.

Proof of Theorem 3.2. It is enough to prove the theorem for any $K = [0, s] \times [-L, L]$, with $0 < s < T$ and $L > 0$. Let us denote by

$$A_\delta(v)(t, x) := a(t, x) \int_{\mathbb{R}} v(t, y) \rho^\delta_x(x - y)\, dy + \int_{\mathbb{R}} v(t, y)\, d(a(t, y) \rho^\delta(x - y)).$$

Fix $0 < s < T$ and $L > 0$. Denote by $K := [0, s] \times [-L, L]$, $K^* := [0, s] \times [-L-1, L+1]$. Recall that $\Omega := [0, T] \times \mathbb{R}$. The proof of Theorem 3.2 will use the following lemma.

Lemma 6.1. *Let* $v \in C(\Omega)$. *Then there is a constant* c, *such that for any* $0 < \delta < 1$

$$\|A_\delta(v)\|_{L^1(K)} \le c\|v\|_{C(K^*)}.$$

Moreover,

$$A_\delta(v) \longrightarrow 0 \text{ in } L^1(K) \text{ as } \delta \longrightarrow 0. \tag{6.1}$$

Proof. We split A_δ into two parts, $A_\delta(v) = I^1_\delta(v) + I^2_\delta(v)$, where

$$I^1_\delta(v)(t,x) := \int_{\mathbb{R}} v(t,y)\rho^\delta(x-y)\,da(t,y),$$

and

$$I^2_\delta(v)(t,x) := \int_{\mathbb{R}} v(t,y)(a(t,x)-a(t,y))\rho^\delta_x(x-y)\,dy.$$

We proceed with an estimate for $I^1_\delta(v)$. We have

$$\begin{aligned}
\|I^1_\delta(v)\|_{L^1(K)} &= \int_0^s \int_{-L}^{L} |\int_{\mathbb{R}} v(t,y)\rho^\delta(x-y)\,da(t,y)|\,dxdt \\
&\le \int_0^s \int_{-L}^{L} \int_{-L-1}^{L+1} |v(t,y)|\rho^\delta(x-y)\,d|a(t,y)|\,dxdt \\
&\le \|v\|_{C(K^*)} \int_0^s \int_{-L-1}^{L+1} \int_{-L}^{L} \rho^\delta(x-y)\,dxd|a(t,y)|\,dt \\
&\le \|v\|_{C(K^*)} \int_0^s \int_{-L-1}^{L+1} d|a(t,y)|dt = \|v\|_{C(K^*)} \int_0^s Var_{[-L-1,L+1]}a(t,\cdot)\,dt.
\end{aligned}$$

From (2.4) it follows that

$$\|I^1_\delta(v)\|_{L^1(K)} \le c\|v\|_{C(K^*)},$$

where c is independent of δ.

Now we continue with $I^2_\delta(v)$. Using the definition of ρ^δ, we derive

$$\int_{-L}^{L} |I^2_\delta|\,dx \le \|v\|_{C(K^*)} \int_{-L}^{L} \int_{-L-1}^{L+1} |a(t,x)-a(t,y)|\frac{1}{\delta^2}|\rho_x(\frac{x-y}{\delta})|\,dydx.$$

After a substitution $z = \frac{x-y}{\delta}$, we obtain

$$\int_{-L}^{L} |I^2_\delta|\,dx \le c\|v\|_{C(K^*)} \int_{-L}^{L} \frac{1}{\delta} \int_{-1}^{1} |a(t,x)-a(t,x-\delta z)|\,dzdx$$

$$= c\|v\|_{C(K^*)}\int_{-1}^{1}\frac{1}{\delta}\int_{-L}^{L}|a(t,x)-a(t,x-\delta z)|\,dxdz.$$

Condition (2.2) implies that the values of $a(t,\cdot)$ at the points of jump should be between the left and right limit of the function $a(t,\cdot)$ at the jump. In this setting (see Lemma 9.2, [4], page 53), for $|z|\le 1$, we derive

$$\int_{-L}^{L}|a(t,x)-a(t,x-\delta z)|\,dxdz \le \delta Var_{[-L-\delta,L+\delta]}a(t,\cdot).$$

Hence

$$\int_{-L}^{L}|I_\delta^2|\,dx \le c\|v\|_{C(K^*)}Var_{[-L-\delta,L+\delta]}a(t,\cdot) \le c\|v\|_{C(K^*)}Var_{[-L-1,L+1]}a(t,\cdot).$$

Then from estimate (2.4), we have

$$\|I_\delta^2(v)\|_{L^1(K)} \le c\|v\|_{C(K^*)}\int_0^s Var_{[-L-1,L+1]}a(t,\cdot)\,dt \le c\|v\|_{C(K^*)}.$$

Combining these estimates, we get that $\|A_\delta(v)\|_{L^1(K)} \le c\|v\|_{C(K^*)}$, where the constant c is independent of δ and v. This proves the first part of the lemma.

In order to complete the proof, it is enough to show that $\|A_\delta(v)\|_{L^1(K)} \longrightarrow 0$ for smooth v. The general case follows by density arguments. But when v is smooth, $a(v*\rho^\delta)_x \longrightarrow av_x$ and $(av_x)*\rho^\delta \longrightarrow av_x$ in $L^1(K)$ as $\delta \longrightarrow 0$. □

Now, we continue with the proof of Theorem 3.2. We use the notation introduced in Section 3. Recall that $\Omega := [0,T]\times \mathbb{R}$. For any $g\in L^1_{loc}(\Omega)$ we define the operator

$$S_{\epsilon,\delta}g(t,x) := g*\eta^\epsilon(t)*\rho^\delta(x).$$

Note that, for $\epsilon < T-s$, $S_{\epsilon,\delta}v$ is well defined on K. When $\epsilon=0$, we mean that $S_{0,\delta}g := g*\rho^\delta$. Similarly, when $\delta=0$, $S_{\epsilon,0}g := g*\eta^\epsilon$. Recall that $v\in C(\Omega)$, $v(0,x)=0$ and $v^\epsilon := S_{\epsilon,\epsilon}v$, $v^{\epsilon,0} := S_{\epsilon,0}v$, $v^{0,\epsilon} := S_{0,\epsilon}v$, $r^\epsilon := v_t^\epsilon + av_x^\epsilon$. With this notation we have

$$r^\epsilon = v_t^\epsilon + av_x^\epsilon - A_\epsilon(v^{\epsilon,0}) + A_\epsilon(v^{\epsilon,0}) =: \Delta^\epsilon + A_\epsilon(v^{\epsilon,0}).$$

But $A_\epsilon(v^{\epsilon,0}) = A_\epsilon(v^{\epsilon,0}-v) + A_\epsilon(v)$ and by Lemma 6.1

$$\|A_\epsilon(v^{\epsilon,0})\|_{L^1(K)} \le c\|v^{\epsilon,0}-v\|_{C(K^*)} + \|A_\epsilon(v)\|_{L^1(K)}.$$

Again by Lemma 6.1, we have $\|A_\epsilon(v)\|_{L^1(K)} \longrightarrow 0$. Also, from the definition of $v^{\epsilon,0}$, it follows that $\|v^{\epsilon,0}-v\|_{C(K^*)} \longrightarrow 0$ as $\epsilon \longrightarrow 0$. We combine the above estimates and derive

$\|A_\epsilon(v^{\epsilon,0})\|_{L^1(K)} \longrightarrow 0$ as $\epsilon \longrightarrow 0$. Thus, we are left with the estimation of $\|\Delta^\epsilon\|_{L^1(K)}$ We can rewrite Δ^ϵ as

$$\Delta^\epsilon(t,x) = \int_0^T \int_{\mathbb{R}} v(\tau,y)\partial_t \eta^\epsilon(t-\tau)\rho^\epsilon(x-y)\,dyd\tau$$

$$-\int_0^T \int_{\mathbb{R}} v(\tau,y)\eta^\epsilon(t-\tau)\,d(a(\tau,y)\rho^\epsilon(x-y))d\tau.$$

For $\epsilon > 0$ and sufficiently small, i.e. $\epsilon < T-s$, the function $\Phi(\tau,y) := \eta^\epsilon(t-\tau)\rho^\epsilon(x-y)$ is in $C^\infty(\Omega)$ and with compact support in $[0,T) \times \mathbb{R}$. Hence, using the definition of a weak solution, we find

$$\Delta^\epsilon(t,x) = \int_{\mathbb{R}} v(0,y)\Phi(0,y)\,dy.$$

Because $v(0,y) = 0$ for any $y \in \mathbb{R}$, we obtain $\Delta^\epsilon(t,x) = 0$ for any $(t,x) \in K$. Therefore $r^\epsilon = \Delta^\epsilon + A_\epsilon(v^{\epsilon,0}) \longrightarrow 0$ in $L^1(K)$ as $\epsilon \longrightarrow 0$. □

Proof of Lemma 3.4. We will reformulate the lemma for the forward problem, because the notation is simpler and the proof goes similarly to Lemma 4.1.1 in [1]. We make a change of variables $t := s - t$ and instead of changing the name of the function a, we change the condition (2.2), imposed on a. In this setting the one-sided Lipschitz condition (2.2) is reformulated as

$$(a(t,x) - a(t,y))(x-y) \le m(t)(x-y)^2, \text{ for all } x, y \in \mathbb{R}, \tag{6.3}$$

where (without loss of generality) we assume $m(t) > 0$. The condition (2.1) stays the same. Here e^ϵ will be the solution to the problem

$$e^\epsilon_t + a^\epsilon e^\epsilon_x = \phi^\epsilon, \quad (t,x) \in (0,s) \times \mathbb{R}, \tag{6.4}$$

$$e^\epsilon(0,x) = 0, \quad x \in \mathbb{R}, \tag{6.5}$$

with $a^\epsilon := S_{0,\epsilon^2} a$ and $\phi^\epsilon \longrightarrow 0$ in $L^1_{loc}([0,s] \times \mathbb{R})$ as $\epsilon \to 0$. Then, Lemma 3.4 is equivalent to

Lemma 6.2. *Let a satisfy* (2.1) *and* (6.3) *and* e^ϵ *be the solution of the problem* (6.4)-(6.5), *where* $\phi^\epsilon \longrightarrow 0$ *in* $L^1_{loc}([0,s] \times \mathbb{R})$. *Then*

$$\|e^\epsilon(s,\cdot)\|_{L^1_{loc}(\mathbb{R})} \to 0 \text{ as } \epsilon \to 0.$$

Proof. For the proof of the lemma we only need

$$\|a(t,\cdot)\|_{L^\infty(\mathbb{R})} \le m(t) \tag{6.6}$$

with $m \in L^1[0,T]$. Note that, (6.6) is a weaker assumption than (2.1). It is easy to verify that if a satisfies (6.3) and (6.6), then

$$a_x^\epsilon(t,x) \leq m(t), \text{ for any } x \in \mathbb{R}, \tag{6.7}$$

and

$$\|a^\epsilon(t,\cdot)\|_{L^\infty(\mathbb{R})} \leq m(t). \tag{6.8}$$

Let us define $\psi^\epsilon(t,x) := |e^\epsilon(t,x)| \exp^{-\int_0^t m(\tau)\,d\tau}$. Then, since m is a nonnegative integrable function, it is enough to show that $\psi^\epsilon(s,\cdot) \longrightarrow 0$ in $L^1_{loc}(R)$. Using (6.4), we get

$$\psi_t^\epsilon + (a^\epsilon \psi^\epsilon)_x = \phi_1^\epsilon, \tag{6.9}$$

with

$$\phi_1^\epsilon = (|e^\epsilon|_t + a^\epsilon |e^\epsilon|_x) exp^{-\int_0^t m(\tau)\,d\tau} + (a_x^\epsilon - m)|e^\epsilon| exp^{-\int_0^t m(\tau)\,d\tau} :=$$

$$\phi_2^\epsilon + (a_x^\epsilon - m)|e^\epsilon| exp^{-\int_0^t m(\tau)\,d\tau}.$$

Because of (6.7), we obtain that

$$\psi_t^\epsilon + (a^\epsilon \psi^\epsilon)_x \leq \phi_2^\epsilon. \tag{6.10}$$

Here $\phi_2^\epsilon \longrightarrow 0$ in $L^1_{loc}([0,s] \times \mathbb{R})$, since $\phi^\epsilon \longrightarrow 0$ in $L^1_{loc}([0,s] \times \mathbb{R})$.

The remainder of the proof is essentially an application of Green's theorem for the left-hand side of (6.10). However, since we have no reference for the generality we need, we shall provide the details independently.

Fix $x_0 \in \mathbb{R}$, $s > 0$ and $R > 0$. Let us define $M(t) := \int_t^s m(\tau) d\tau$,

$$\Omega_s := \{(t,x) \ : \ 0 \leq t \leq s \text{ and } |x - x_0| \leq R + M(t)\},$$

and integrate (6.10) over Ω_s. We get

$$\int\int_{\Omega_s} \psi_t^\epsilon dx dt + \int\int_{\Omega_s} (a^\epsilon \psi^\epsilon)_x dx dt \leq \int\int_{\Omega_s} \phi_2^\epsilon dt dx, \tag{6.11}$$

where $\int\int_{\Omega_s} \phi_2^\epsilon dt dx \longrightarrow 0$ as $\epsilon \to 0$.

The idea of the proof is to split the domain Ω_s into three parts, integrate the two terms in the left hand side of (6.11) on each of the three domains and use the estimate (6.8) for the velocity a^ϵ.We divide Ω_s in three regions

$$\Omega_s^0 := \{(t,x) \ : \ (t,x) \in \Omega_s, \ |x - x_0| \leq R\},$$

$$\Omega_s^1 := \{(t,x) \ : \ (t,x) \in \Omega_s, \ x - x_0 > R\},$$

$$\Omega_s^2 := \{(t,x) \ : \ (t,x) \in \Omega_s, \ x - x_0 < -R\}.$$

Then

$$\int\int_{\Omega_s} \psi_t^\epsilon + (a^\epsilon \psi^\epsilon)_x dxdt = \sum_{i=0}^{2} \int\int_{\Omega_s^i} \psi_t^\epsilon + (a^\epsilon \psi^\epsilon)_x dxdt.$$

Denote $I_s^i := \int\int_{\Omega_s^i} \psi_t^\epsilon dxdt$, for $i = 1, 2$. We represent Ω_s^1 and Ω_s^2 as

$$\Omega_s^1 := \{(t,x) \ : \ R < x - x_0 < R + M(s) \ , \ 0 \le t \le M^{-1}(x - x_0 - R)\},$$

$$\Omega_s^2 := \{(t,x) \ : \ -R - M(s) < x - x_0 < -R \ , \ 0 \le t \le M^{-1}(-x + x_0 - R)\},$$

where M^{-1} is the inverse function of M (M is strictly decreasing). Then we integrate by parts and after change of variables $y = x - x_0 - R$, we obtain

$$I_s^1 = \int_0^{M(s)} \psi^\epsilon(M^{-1}(y), y + x_0 + R)dy - \int_0^{M(s)} \psi^\epsilon(0, y + x_0 + R)dy.$$

We have $\int_0^{M(0)} \psi^\epsilon(0, y+x_0+R)dy = 0$ because of (6.5), and again after a change of variables $z = M^{-1}(y)$, we conclude

$$I_s^1 = \int_0^s \psi^\epsilon(z, x_0 + R + M(z))m(z)dz. \tag{6.12}$$

Similarly, we obtain

$$I_s^2 = \int_0^s \psi^\epsilon(z, x_0 - R - M(z))m(z)dz. \tag{6.13}$$

Now, we consider $\int\int_{\Omega_s} (a^\epsilon \psi^\epsilon)_x dxdt$. After integration by parts, we have

$$\begin{aligned} \int\int_{\Omega_s} (a^\epsilon \psi^\epsilon)_x dxdt = &\int_0^s a^\epsilon(z, x_0 + R + M(z))\psi^\epsilon(z, x_0 + R + M(z))dz \\ &- \int_0^s a^\epsilon(z, x_0 - R - M(z))\psi^\epsilon(z, x_0 - R - M(z))dz. \end{aligned} \tag{6.14}$$

Combining (6.12), (6.13) and (6.14), we get

$$\int\int_{\Omega_s^0} \psi_t^\epsilon dxdt + r_1^\epsilon + r_2^\epsilon = \int\int_{\Omega_s} \psi_t^\epsilon + (a^\epsilon \psi^\epsilon)_x dxdt \le \int\int_{\Omega_s} \phi_2^\epsilon dtdx, \tag{6.15}$$

where $r_1^\epsilon := \int_0^s (m(z) + a^\epsilon(z, x_0 + R + M(z)))\psi^\epsilon(z, x_0 + R + M(z))dz$ and $r_2^\epsilon := \int_0^s (m(z) - a^\epsilon(z, x_0 - R - M(z)))\psi^\epsilon(z, x_0 - R - M(z))dz$. We know that ψ^ϵ is non-negative and a^ϵ satisfies (6.8), therefore we have that $r_i \ge 0$ for $i = 1, 2$. Then from (6.15) we have

$$\int_{|x - x_0| < R} \psi^\epsilon(s, x)dx = \int\int_{\Omega_s^0} \psi_t^\epsilon dxdt \le \int\int_{\Omega_s} \phi_2^\epsilon dtdx.$$

Hence, $\|\psi^\epsilon(s, \cdot)\|_{L^1_{loc}(\mathbb{R})} \longrightarrow 0$ and the proof is completed. □

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