

Linear Transport Equations with μ -Monotone Coefficients

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We introduce the concept of a μ -monotone function. It allows us to extend the existing theory for Filippov solutions to ODE, linear transport equations, and conservation laws for a wider range of transport velocities (A_1, \dots, A_d) and fluxes (f_1, \dots, f_d) . © 2001 Academic Press

1. INTRODUCTION

Linear transport equations with non-smooth coefficients, while interesting in their own right, occur in the study of nonlinear transport equations such as conservation laws

$$u_t + \sum_{i=1}^d \frac{\partial}{\partial x_i} f_i(u) = 0, \quad (\text{CL})$$

$$u(0, \mathbf{x}) = u_0(\mathbf{x}). \quad (\text{IC})$$

For example, the uniqueness theory for (CL)–(IC) is closely related to the existence theory of the adjoint problem (AP), that is, the linear transport



equation

$$\varphi_t + \sum_{i=1}^d A_i \varphi_{x_i} = D\varphi + C, \quad (\text{AP})$$

with velocities

$$A_i(t, \mathbf{x}) := \frac{f_i(t, \mathbf{x}, u_1(t, \mathbf{x})) - f_i(t, \mathbf{x}, u_2(t, \mathbf{x}))}{u_1(t, \mathbf{x}) - u_2(t, \mathbf{x})},$$

and initial condition $\varphi(T, \mathbf{x}) = \varphi_T(\mathbf{x})$ (u_1 and u_2 are solutions to (CL)–(IC)).

In the context of understanding the analytical properties of the solution to the conservation law (CL)–(IC), linear transport equations have been studied by Conway [4], Tadmor [13], Perthame [6], Bouchut and James [2], Poupand and Rascle [12], and others. The main difficulty in their analysis is the fact that the transport velocities A_i will, in general, not be continuous (since the solution u to (CL)–(IC) develops shocks) and therefore the classical theory cannot be applied.

In the one-dimensional case the transport velocities arising from conservation laws, while not continuous, do satisfy the assumption

$$(a(t, x) - a(t, y))(x - y) \geq -K(t)(x - y)^2, \quad K \in L^1[0, T], \quad (\text{OSL})$$

which is called a one-sided Lipschitz condition. The existing theory is sufficient to treat all transport equations associated with conservation laws in this case.

In more than one space dimension, there is not yet a suitable classification of the discontinuities of the velocities of the linear transport equations associated with conservation laws. A step along the way is to introduce conditions on the transport velocities which, when used in the place of (OSL), allow a natural extension of the one space dimension results to several space dimensions. Such a condition on A , namely

$$(\mathbf{x} - \mathbf{y}) \cdot (A(t, \mathbf{x}) - A(t, \mathbf{y})) \geq -K(t) |\mathbf{x} - \mathbf{y}|^2, \quad K \in L^1[0, T], \quad (\text{F})$$

was proposed by Filippov in [7] and was used to develop uniqueness-stability results for ODEs with possibly discontinuous coefficients. These results have been the cornerstone for the study of linear transport equations with non-smooth velocities and have been utilized by several authors (Conway [4], Dolcetta and Perthame [6], and Poupand and Rascle [12]) to develop an existence-uniqueness-stability theory for transport equations. However, this theory does not apply to a wide range of problems and treats only equations with velocities that satisfy condition (F).

In this paper, we propose a family of sufficient conditions, called μ -monotonicity, which can replace Filippov's condition (F). We present examples showing that the class of μ -monotone functions is, in general,

different from (and larger than) Filippov’s class (F). Therefore, it allows us to develop an existence-uniqueness-stability theory for a wide class of ODEs and linear transport equations.

Special attention is given to the case $\mu(\mathbf{z}) = \|\mathbf{z}\|_p^p, 1 \leq p < \infty$, which is called p -monotonicity (the case $p = 2$ corresponds to Filippov’s condition (F)).

We derive (see Section 4) that an ODE with a μ -monotone right-hand side has a unique on the left Filippov solution. Then, in the next section, we show the existence of a Lipschitz solution to the Cauchy problem for the nonconservative linear transport equation (AP) in the case of p -monotone velocities A . Under additional assumptions on A , this condition guarantees that the constructed solution is stable under smoothing of the coefficients. Finally, in Section 6, we apply our results to derive a uniqueness theorem for weak solutions to conservation laws. Particular cases of our assumption on the flux (see condition (6.1)) have been considered by Conway [4] and Perthame [6].

One-sided uniqueness of the Filippov characteristics was used in [2, 12] to develop an existence-uniqueness theory for measure solutions to linear transport equations in conservative form. The authors used condition (F) to obtain stability results for these solutions. It is possible to generalize the results of [12] to p -monotone velocities for $1 \leq p < \infty$. However, we shall not treat this case since we shall only consider the nonconservative problem.

2. PRELIMINARIES AND DEFINITIONS

For $\mathbf{x} = (x_1, \dots, x_d)$ in \mathbb{R}^d , we let

$$\|\mathbf{x}\|_p := (|x_1|^p + \dots + |x_d|^p)^{1/p}, \quad 1 \leq p < \infty,$$

be the ℓ_p norm on \mathbb{R}^d , and for $p = 2$ we write $|\mathbf{x}| := \|\mathbf{x}\|_2$. We denote by

$$\mathbf{x} \cdot \mathbf{y} := \sum_{i=1}^d x_i y_i$$

the inner product of $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$. Further, we write (t, \mathbf{x}) for the point $(t, x_1, \dots, x_d) \in \mathbb{R} \times \mathbb{R}^d$ and G for the set $[0, T] \times \mathbb{R}^d, T > 0$.

Let $\mu : \mathbb{R}^d \rightarrow \mathbb{R}$ be a non-negative function that satisfies the Lipschitz condition

$$|\mu(\mathbf{z}') - \mu(\mathbf{z}'')| \leq M|\mathbf{z}' - \mathbf{z}''|, \tag{L}$$

with an absolute constant $M > 0$, and $\mu(\mathbf{z}) = 0$ iff $\mathbf{z} = 0$. We define the following two sets of indices

$$I := \left\{ i : \frac{\partial \mu}{\partial z_i} \text{ is continuous in } \mathbb{R}^d \right\}, \quad J = \{1, \dots, d\} \setminus I.$$

DEFINITION 2.1. We say that a vector valued measurable function $A := (A_1, \dots, A_d)$, $A : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$, is μ -monotone, if there is a non-negative function $K \in L^1[0, T]$, such that for almost every $t \in (0, T)$

$$\sum_{i \in \mathcal{D}} \frac{\partial \mu}{\partial z_i}(\mathbf{x} - \mathbf{y}) [A_i(t, \mathbf{x}) - A_i(t, \mathbf{y})] \geq -K(t)\mu(\mathbf{x} - \mathbf{y}), \quad \mathcal{D} = I, J, \quad (*)$$

for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$.

A special interest to us is the case $\mu(\mathbf{z}) = \|\mathbf{z}\|_p^p$, $1 \leq p < \infty$. Then we simply say that A is p -monotone. In this case $I = \{1, \dots, d\}$ for $1 < p < \infty$ and $I = \emptyset$ for $p = 1$. Therefore, condition (*) can be written as

$$\nabla_z \mu(\mathbf{x} - \mathbf{y}) \cdot [A(t, \mathbf{x}) - A(t, \mathbf{y})] \geq -K(t)\mu(\mathbf{x} - \mathbf{y}).$$

Notice that the notion of μ -monotonicity allows discontinuities of A and is a generalization of the usual monotonicity in one space dimension. For example, the class of p -monotone functions, $1 \leq p < \infty$, contains all vector functions $A = (A_1(t, x_1), \dots, A_d(t, x_d))$ with $A_i(t, \cdot)$ a non-decreasing function of the x_i variable, $i = 1, \dots, d$.

The introduced concept is of interest mainly in the multi-dimensional case, since for $d = 1$ the p -monotone condition on A , $1 \leq p < \infty$, coincides with (OSL). For $d > 1$ condition (*) introduces a new class of functions, which is different from the class of transport velocities that satisfy the Filippov condition (F) (see the examples below). In fact, only the class of 2-monotone functions coincides with the Filippov class (F).

EXAMPLE 2.1. We consider $a : [0, T] \times \mathbb{R}^2 \rightarrow \mathbb{R}$, defined by

$$a(t, x_1, x_2) := \begin{cases} \operatorname{sgn} x_1, & \text{if } |x_1| \geq t, \\ x_1/t, & \text{if } |x_1| < t. \end{cases}$$

Then $A(t, x_1, x_2) := (a(t, x_1, x_2), a(t, x_1, x_2))$ is a 1-monotone function that does not satisfy condition (F).

EXAMPLE 2.2. The function

$$A(t, x_1, x_2) = (\operatorname{sgn}(x_1 + rx_2), r \operatorname{sgn}(x_1 + rx_2))$$

satisfies condition (F) for all $r \in \mathbb{R}$, but is not 1-monotone when $r > 1$.

EXAMPLE 2.3. The function

$$\begin{aligned} &A(t, x_1, x_2, x_3, x_4) \\ &= (a(t, x_1, x_2), a(t, x_1, x_2), \operatorname{sgn}(x_3 + rx_4), r \operatorname{sgn}(x_3 + rx_4)), \end{aligned}$$

is μ -monotone with $\mu(\mathbf{z}) = |z_1| + |z_2| + (z_3^2 + z_4^2)/2$, but does not satisfy condition (F) when $r > 1$.

3. FILIPPOV'S THEORY FOR ODES WITH DISCONTINUOUS RIGHT-HAND SIDE

In this section, we recall some results of Filippov on ODEs (see [7]) which we use later in this paper.

The vector function A is said to satisfy condition (B) in $G := [0, T] \times \mathbb{R}^d$, $T > 0$, if it is defined almost everywhere in G , is measurable, and for every bounded closed domain $D \subset G$, there is a summable function $B(t)$, such that for almost every (t, \mathbf{y}) in D

$$|A(t, \mathbf{y})| \leq B(t). \tag{B}$$

For a measurable function f on \mathbb{R}^d we define

$$M_{\mathbf{x}}\{f\} := \lim_{\delta \rightarrow 0} \text{ess sup}_{\mathbf{x}' \in U(\mathbf{x}, \delta)} f(\mathbf{x}'),$$

where $U(\mathbf{x}, \delta)$ is a δ -neighborhood of the point $\mathbf{x} \in \mathbb{R}^d$ with respect to the Euclidean norm. Similarly we define $m_{\mathbf{x}}\{f\}$ with *ess sup* replaced by *ess inf*.

We are interested in the theory of existence of a solution to the linear transport equation

$$\varphi_t + \sum_{i=1}^d A_i(t, \mathbf{x})\varphi_{x_i} = D(t, \mathbf{x})\varphi + C(t, \mathbf{x}), \tag{3.1}$$

$$\varphi(0, \mathbf{x}) = \varphi_0(\mathbf{x}), \tag{3.2}$$

with μ -monotone transport velocities $A(t, \mathbf{x})$. This theory is closely related to the study of Filippov solutions to the system of ODEs

$$\frac{d\chi}{ds} = A(s, \chi), \tag{3.3}$$

$$\chi(t; (t, \mathbf{x})) = \mathbf{x}, \tag{3.4}$$

with possibly discontinuous right-hand side A .

DEFINITION 3.1. A vector function $\chi(s) := \chi(s; (t, \mathbf{x}))$, defined on the interval (t_1, t_2) , is called a Filippov solution to (3.3)–(3.4), if

- (i) it is absolutely continuous,
- (ii) $\chi(t) = \mathbf{x}$,
- (iii) for almost all $s \in (t_1, t_2)$

$$m_{\chi(s)}\{A_i(s, \cdot)\} \leq \chi'_i(s) \leq M_{\chi(s)}\{A_i(s, \cdot)\}, \quad i = 1, \dots, d,$$

where $M_{\chi(s)}$ and $m_{\chi(s)}$ are applied to the space variable with s fixed.

We shall refer to a Filippov solution to (3.3)–(3.4) as a *Filippov generalized characteristic*.

Necessary and sufficient conditions for a function χ to be a Filippov solution to (3.3)–(3.4) are given by the next lemma (see [7, Lemma 2]).

LEMMA 3.1. *The absolutely continuous vector function $\chi(s) := \chi(s; (t, \mathbf{x}))$ is a Filippov solution to (3.3)–(3.4) if and only if for almost all s and for every vector ϑ the following inequality holds*

$$\vartheta \cdot \frac{d\chi(s)}{ds} \leq M_{\chi(s)} \{A(s, \cdot) \cdot \vartheta\}.$$

Remark 3.1. The above mentioned inequality can be viewed also as

$$\vartheta \cdot \frac{d\chi(s)}{ds} \geq m_{\chi(s)} \{A(s, \cdot) \cdot \vartheta\}.$$

Let us make the observation (see [7, Theorem 5]) that under certain conditions on A any Filippov solution to (3.3)–(3.4) is a global solution. Namely, the following remark is true.

Remark 3.2. If the velocity A satisfies condition (B) in G , then every Filippov generalized characteristic is defined on $[0, T]$.

In the next section we prove that one-sided uniqueness of the Filippov solution to (3.3)–(3.4) holds whenever the transport velocity A is μ -monotone. In Section 5, we apply these new results to linear transport equations. For this, we shall utilize a limiting argument using mollifiers. To describe this procedure, let φ^h be the unique solution to the smooth problem

$$\varphi_i^h + \sum_{i=1}^d A_i^h(t, \mathbf{x}) \varphi_{x_i}^h = D^h(t, \mathbf{x}) \varphi^h + C^h(t, \mathbf{x}), \quad (3.5)$$

$$\varphi^h(0, \mathbf{x}) = \varphi_0^h(\mathbf{x}), \quad (3.6)$$

where A_i^h, C^h, D^h are mollifications (via convolution with a positive kernel with respect to the variable (t, \mathbf{x})) of A_i, C , and D (we extend these functions to be 0 outside of G). More precisely,

$$A_i^h(t, \mathbf{x}) := (A_i * \zeta^h)(t, \mathbf{x}),$$

$$i = 1, \dots, d, \text{ where } \zeta^h(t, \mathbf{x}) := \frac{1}{h^{d+1}} \zeta\left(\frac{t}{h}, \frac{\mathbf{x}}{h}\right),$$

and ζ is a C^∞ non-negative function that is supported on $\{(t, \mathbf{x}) \in \mathbb{R} \times \mathbb{R}^d : |(t, \mathbf{x})| \leq 1\}$ with $\int_{\mathbb{R}^{d+1}} \zeta(t, \mathbf{x}) dt d\mathbf{x} = 1$. Also, we have that

$$\varphi_0^h(\mathbf{x}) := (\varphi_0 * \xi)(\mathbf{x}) \quad \text{with } \xi^h(\mathbf{x}) := \frac{1}{h^d} \xi\left(\frac{\mathbf{x}}{h}\right),$$

where $\xi = \xi(\mathbf{x})$ is a C^∞ non-negative function, supported on the ball $\{\mathbf{x} \in \mathbb{R}^d : |\mathbf{x}| \leq 1\}$ with $\int_{\mathbb{R}^d} \xi(\mathbf{x}) d\mathbf{x} = 1$.

It is known (see [4]) that the solution φ^h to (3.5)–(3.6) can be expressed explicitly in the form

$$\begin{aligned} \varphi^h(t, \mathbf{x}) &= \varphi_0^h(\chi^h(0; (t, \mathbf{x}))) \exp \int_0^t D^h(\tau, \chi^h(\tau; (t, \mathbf{x}))) d\tau \\ &\quad + \int_0^t C^h(s, \chi^h(s; (t, \mathbf{x}))) \exp \int_s^t D^h(\tau, \chi^h(\tau; (t, \mathbf{x}))) d\tau ds, \end{aligned} \tag{3.7}$$

where $\chi^h(s; (t, \mathbf{x}))$ is the solution of the smooth system

$$\frac{d\chi^h}{ds} = A^h(s, \chi^h), \tag{3.8}$$

$$\chi^h(t; (t, \mathbf{x})) = \mathbf{x}. \tag{3.9}$$

To prove the stability of the solution to (3.1)–(3.2) under any mollification with respect to (t, \mathbf{x}) , Conway (see [3, 4]) introduces another notion of a solution to (3.3)–(3.4)—the so-called *Full solution*.

DEFINITION 3.2. A function $\chi(s) := \chi(s; (t, \mathbf{x}))$, defined on (t_1, t_2) , is called a Full solution to (3.3)–(3.4) if

- (i) it is absolutely continuous,
- (ii) $\chi(t) = \mathbf{x}$,
- (iii) for almost all s in (t_1, t_2)

$$m_{(s, \chi(s))}\{A_i\} \leq \chi'_i(s) \leq M_{(s, \chi(s))}\{A_i\}, \quad i = 1, \dots, d.$$

Here $m_{(s, \chi(s))}$ and $M_{(s, \chi(s))}$ are applied to A_i as a function of both space and time.

As is shown in [4], the theory of Full solutions to (3.3)–(3.4) is identical with the theory of Filippov solutions to the autonomous system

$$\frac{d\bar{\chi}}{ds} = A(\bar{\chi}), \tag{3.10}$$

$$\bar{\chi}(t; (t, \mathbf{x})) = (x_1, \dots, x_d, t),$$

where $\bar{\chi} = (\chi_1, \dots, \chi_d, s)$ and

$$A(\bar{\chi}) = (A_1(s, \chi), \dots, A_d(s, \chi), 1). \tag{3.11}$$

Because of this, the results in [7], concerning Filippov solutions to (3.10) can be stated for Full solutions to problem (3.3)–(3.4). In particular, the following lemma (see [7, Corollary 2 of Theorem 11]) is valid and will be a key argument in our stability results.

LEMMA 3.2. *If A is bounded and problem (3.3)–(3.4) has a unique Full solution $\chi(s; (t, \mathbf{x}))$ on $[0, t]$, then*

$$\lim_{h \rightarrow 0} \chi^h(s; (t, \mathbf{x})) = \chi(s; (t, \mathbf{x})), \quad (3.12)$$

uniformly on $0 \leq s \leq t$.

We would like to mention that the assumption of uniqueness of the Full solution to (3.3)–(3.4) is stronger than the same assumption for the Filippov solution to (3.3)–(3.4) (see [4, Lemma 3]). We need stronger conditions on A to guarantee the uniqueness of the Full solution to (3.3)–(3.4) and therefore the stability under any mollification of the solution to (3.1)–(3.2). The above lemma can be stated under the assumption of uniqueness of the Filippov solution to (3.3)–(3.4), but then we have to use a special type of mollification to define A^h and to get χ^h . For details, see [11].

In Section 6, we discuss the uniqueness theory of weak solutions to quasi-linear conservation laws

$$u_t + \sum_{i=1}^d \frac{\partial}{\partial x_i} f_i(t, \mathbf{x}, u) + g(t, \mathbf{x}, u) = 0, \quad (3.13)$$

$$u(0, \mathbf{x}) = u_0(\mathbf{x}). \quad (3.14)$$

Here $f_i(t, \mathbf{x}, \cdot), g(t, \mathbf{x}, \cdot) \in C^1(\mathbb{R})$ for each fixed $(t, \mathbf{x}) \in G$, and when u is restricted on a bounded interval of the real line, f_i, g , and their derivatives are bounded and measurable in G .

A function u is said to be a weak solution to (3.13)–(3.14) if it is bounded and

$$\begin{aligned} & \int_{\mathbb{R}^d} \int_0^T \left\{ u \varphi_t + \sum_{i=1}^d f_i(t, \mathbf{x}, u) \varphi_{x_i} - g(t, \mathbf{x}, u) \varphi \right\} dt d\mathbf{x} \\ & + \int_{\mathbb{R}^d} u_0(\mathbf{x}) \varphi(0, \mathbf{x}) d\mathbf{x} = 0, \end{aligned} \quad (3.15)$$

for all φ which are locally Lipschitz continuous and have compact support in G . Weak solutions are not unique, but uniqueness of weak solutions that satisfy auxiliary entropy conditions was shown by Oleinik, Conway, Vol’pert, and Kruzhkov. Generally, the uniqueness theory of weak solutions to (3.13)–(3.14) is closely related to the theory of existence of solutions to linear transport equations (3.1)–(3.2). We follow the approach in [4] and use the results from Section 5 to introduce new entropy conditions that extend the ones discussed in [4, 6].

4. ON THE EXISTENCE AND UNIQUENESS OF THE FILIPPOV SOLUTION

First we shall need some lemmas and definitions (see [8, pp. 142, 144]).

LEMMA 4.1. *Let $\mu : \mathbb{R}^d \rightarrow \mathbb{R}$ be a Lipschitz function with continuous derivatives. If $\gamma(s) = (\gamma_1(s), \dots, \gamma_d(s))$ denotes an absolutely continuous vector function on $(0, T)$, then the function $K : (0, T) \rightarrow \mathbb{R}$ given by $K(s) = \mu(\gamma(s))$ is absolutely continuous and*

$$\frac{d}{ds}K = \sum_{i=1}^d \frac{\partial \mu}{\partial z_i}(\gamma) \cdot \frac{d}{ds}\gamma_i \quad \text{a.e. in } (0, T). \quad (\#)$$

We define the sgn function by

$$\text{sgn}(x) := \begin{cases} 1, & \text{if } x > 0, \\ 0, & \text{if } x = 0, \\ -1, & \text{if } x < 0. \end{cases}$$

Then the following lemma holds.

LEMMA 4.2. *If $h : (0, T) \rightarrow \mathbb{R}$ is an absolutely continuous function, then*

$$\frac{d}{ds}|h(s)| = \text{sgn}(h(s)) \frac{d}{ds}h(s) \quad \text{for a.e. } s \in (0, T).$$

DEFINITION 4.1. We call a Lipschitz function $\mu : \mathbb{R}^d \rightarrow \mathbb{R}$, for which the chain rule (#) holds for $\mu(\gamma(s))$, with γ an absolutely continuous vector function, a chain rule function.

It follows from Lemma 4.1 that every Lipschitz function with continuous partial derivatives is a chain rule function. For example, by Lemma 4.1 and Lemma 4.2, $\mu(\mathbf{z}) = \|\mathbf{z}\|_p^p$, $1 \leq p < \infty$, or $\mu(z_1, z_2) = |z_1| + z_2^2$ are chain rule functions.

Notice that condition (L), imposed on μ cannot be relaxed. We shall need later on that $\mu(\gamma_1, \dots, \gamma_d)$ is an absolutely continuous function for every choice of the absolutely continuous functions γ_i . This implies that μ has to be Lipschitz (see [9] in the case $d = 1$).

Now we are ready to state the following generalization of the Filippov uniqueness theorem.

THEOREM 4.1. *Let A satisfy condition (B) and be μ -monotone in G , where μ is a chain rule function. Let for almost all $t \in (0, T)$ the functions $A_i(t, \cdot)$, $i \in J$, be continuous. Then through any interior point (t, \mathbf{x}) of G passes a Filippov solution to (3.3)–(3.4), defined on $[0, T]$, and this solution is uniquely determined on $[0, t]$.*

Proof. The function A satisfies condition (B) and from Theorems 4 and 5 in [7] it follows that through any interior point (t, \mathbf{x}) of G passes a Filippov solution of (3.3)–(3.4), defined on the entire interval $[0, T]$.

Fix $0 < t_0 < t$. Assume that χ and ψ are two different Filippov solutions on $[t_0, t]$ that pass through (t, \mathbf{x}) . Let $\gamma(s) := \chi(s) - \psi(s)$. Then, since μ is a Lipschitz function, $\mu(\gamma(s))$ will be an absolutely continuous function defined on $[t_0, t]$. Moreover,

$$\mu(\gamma(t)) = 0, \quad (4.1)$$

and γ is not identically zero on $[t_0, t]$. We shall show that

$$\frac{d}{ds}\mu(\gamma(s)) \geq -2K(s)\mu(\gamma(s)) \quad \text{a.e. in } [t_0, t]. \quad (4.2)$$

Then from (4.1), (4.2), the facts that $\mu \geq 0$, $\mu(\mathbf{z}) = 0$ iff $\mathbf{z} = 0$, and Gronwall's inequality, we can easily obtain that $\gamma(s) = 0$ on $[t_0, t]$, which contradicts our assumption. Hence $\chi(s) = \psi(s)$ on $[t_0, t]$ for every $0 < t_0 < t$, and the proof is completed.

Now to prove (4.2) we use that μ is a chain rule function. Hence we have that a.e. on $(0, T)$

$$\begin{aligned} \frac{d}{ds}\mu(\gamma(s)) &= \sum_{i=1}^d \frac{\partial \mu}{\partial z_i}(\gamma(s)) \frac{d}{ds}\gamma_i(s) \\ &= \sum_{i=1}^d \frac{\partial \mu}{\partial z_i}(\gamma(s)) \frac{d}{ds}\chi_i(s) - \sum_{i=1}^d \frac{\partial \mu}{\partial z_i}(\gamma(s)) \frac{d}{ds}\psi_i(s). \end{aligned} \quad (4.3)$$

Both χ and ψ are Filippov solutions to (3.3)–(3.4) and from Lemma 3.1 for almost all s and every vector ϑ

$$\begin{aligned} \vartheta \cdot \frac{d\chi(s)}{ds} &\geq m_{\chi(s)}\{A(s, \cdot) \cdot \vartheta\} \quad \text{and} \\ \vartheta \cdot \frac{d\psi(s)}{ds} &\leq M_{\psi(s)}\{A(s, \cdot) \cdot \vartheta\}. \end{aligned} \quad (4.4)$$

Also, since A satisfies condition (B), we can find a summable function B , such that almost everywhere in a closed neighborhood of the arcs $t_0 \leq s \leq t$ of the solutions χ and ψ

$$|A(s, \mathbf{y})| \leq B(s). \quad (4.5)$$

We shall now show that (4.2) holds for every s such that $B(s) < \infty$, (4.4), and (4.5) are true, and $A_i(s, \cdot)$ is continuous for $i \in J$. Fix such $s \in [t_0, t]$ and choose a vector $\vartheta \in \mathbb{R}^d$ with components

$$\vartheta_i = \frac{\partial \mu}{\partial z_i}(\gamma_1(s), \dots, \gamma_d(s)), \quad i = 1, \dots, d.$$

Then by (4.4), for every $\epsilon > 0$ we can find $\delta_0 := \delta_0(\epsilon) > 0$ and points $\mathbf{z}^* := \mathbf{z}^*(\delta) \in U(\psi(s), \delta)$ and $\mathbf{x}^* := \mathbf{x}^*(\delta) \in U(\chi(s), \delta)$ such that for all $0 < \delta < \delta_0$

$$\begin{aligned} \sum_{i=1}^d \vartheta_i \frac{d\chi_i(s)}{ds} &\geq \text{ess} \inf_{\mathbf{x}' \in U(\chi(s), \delta)} \left\{ \sum_{i=1}^d \vartheta_i A_i(s, \mathbf{x}') \right\} - \epsilon \\ &> \sum_{i=1}^d \vartheta_i A_i(s, \mathbf{x}^*) - 2\epsilon, \end{aligned} \tag{4.6}$$

$$\begin{aligned} \sum_{i=1}^d \vartheta_i \frac{d\psi_i(s)}{ds} &\leq \text{ess} \sup_{\mathbf{z}' \in U(\psi(s), \delta)} \left\{ \sum_{i=1}^d \vartheta_i A_i(s, \mathbf{z}') \right\} + \epsilon \\ &< \sum_{i=1}^d \vartheta_i A_i(s, \mathbf{z}^*) + 2\epsilon. \end{aligned} \tag{4.7}$$

Then we obtain

$$\frac{d}{ds} \mu(\gamma(s)) > \sum_{i=1}^d \vartheta_i [A_i(s, \mathbf{x}^*) - A_i(s, \mathbf{z}^*)] - 4\epsilon. \tag{4.8}$$

The right-hand side of (4.8) can be written as

$$\begin{aligned} &\sum_{i=1}^d \frac{\partial \mu}{\partial z_i}(\chi(s) - \psi(s)) [A_i(s, \mathbf{x}^*) - A_i(s, \mathbf{z}^*)] \\ &= \sum_{i \in I} \frac{\partial \mu}{\partial z_i}(\chi(s) - \psi(s)) [A_i(s, \mathbf{x}^*) - A_i(s, \mathbf{z}^*)] \\ &\quad + \sum_{i \in J} \frac{\partial \mu}{\partial z_i}(\chi(s) - \psi(s)) [A_i(s, \mathbf{x}^*) - A_i(s, \mathbf{z}^*)] \\ &:= \sigma_1(\epsilon, \delta) + \sigma_2(\epsilon, \delta). \end{aligned}$$

For the first sum we obtain

$$\begin{aligned} \sigma_1(\epsilon, \delta) &= \sum_{i \in I} \left[\frac{\partial \mu}{\partial z_i}(\chi(s) - \psi(s)) - \frac{\partial \mu}{\partial z_i}(\mathbf{x}^* - \mathbf{z}^*) \right] [A_i(s, \mathbf{x}^*) - A_i(s, \mathbf{z}^*)] \\ &\quad + \sum_{i \in I} \frac{\partial \mu}{\partial z_i}(\mathbf{x}^* - \mathbf{z}^*) [A_i(s, \mathbf{x}^*) - A_i(s, \mathbf{z}^*)] := I_1(\epsilon, \delta) + I_2(\epsilon, \delta). \end{aligned}$$

From (4.5) we have that

$$|I_1(\epsilon, \delta)| \leq 2B(s) \sum_{i \in I} \left| \frac{\partial \mu}{\partial z_i}(\chi(s) - \psi(s)) - \frac{\partial \mu}{\partial z_i}(\mathbf{x}^* - \mathbf{z}^*) \right|.$$

Letting $\epsilon \rightarrow 0$ and $\delta \rightarrow 0$, we have $(\mathbf{x}^* - \mathbf{z}^*) \rightarrow (\chi(s) - \psi(s))$. Since the partial derivatives $\partial \mu / \partial z_i, i \in I$, are continuous, we derive that $I_1(\epsilon, \delta) \rightarrow 0$

as $\epsilon, \delta \rightarrow 0$. The function A satisfies the μ -monotonicity condition (*). Therefore we have

$$I_2(\epsilon, \delta) \geq -K(s)\mu(\mathbf{x}^* - \mathbf{z}^*),$$

and for $\sigma_1(\epsilon, \delta)$ we derive that

$$\begin{aligned} \sigma_1(\epsilon, \delta) &\geq -K(s)\mu(\mathbf{x}^* - \mathbf{z}^*) + I_1(\epsilon, \delta), \quad \text{with} \\ I_1(\epsilon, \delta) &\rightarrow 0 \text{ as } \epsilon, \delta \rightarrow 0. \end{aligned} \quad (4.9)$$

Now we write the second sum as

$$\begin{aligned} \sigma_2(\epsilon, \delta) &= \sum_{i \in J} \frac{\partial \mu}{\partial z_i}(\chi(s) - \psi(s)) [A_i(s, \mathbf{x}^*) - A_i(s, \mathbf{z}^*)] \\ &= \sum_{i \in J} \frac{\partial \mu}{\partial z_i}(\chi(s) - \psi(s)) [A_i(s, \mathbf{x}^*) - A_i(s, \chi(s))] \\ &\quad + \sum_{i \in J} \frac{\partial \mu}{\partial z_i}(\chi(s) - \psi(s)) [A_i(s, \chi(s)) - A_i(s, \psi(s))] \\ &\quad + \sum_{i \in J} \frac{\partial \mu}{\partial z_i}(\chi(s) - \psi(s)) [A_i(s, \psi(s)) - A_i(s, \mathbf{z}^*)] \\ &:= J_1(\epsilon, \delta) + J_2 + J_3(\epsilon, \delta). \end{aligned}$$

We have that

$$\begin{aligned} |J_1(\epsilon, \delta)| &\leq M \sum_{i \in J} |A_i(s, \mathbf{x}^*) - A_i(s, \chi(s))|, \\ |J_3(\epsilon, \delta)| &\leq M \sum_{i \in J} |A_i(s, \psi(s)) - A_i(s, \mathbf{z}^*)|. \end{aligned}$$

Let $\epsilon \rightarrow 0$ and $\delta \rightarrow 0$. Then $\mathbf{x}^* \rightarrow \chi(s)$, $\mathbf{z}^* \rightarrow \psi(s)$, and since $A_i(s, \cdot)$, $i \in J$, is continuous, we get that $J_1(\epsilon, \delta) \rightarrow 0$ and $J_3(\epsilon, \delta) \rightarrow 0$ as $\epsilon, \delta \rightarrow 0$. From the μ -monotonicity of A it follows that

$$\sigma_2(\epsilon, \delta) \geq -K(s)\mu(\chi(s) - \psi(s)) + J_1(\epsilon, \delta) + J_3(\epsilon, \delta),$$

where $J_1(\epsilon, \delta), J_3(\epsilon, \delta) \rightarrow 0$ as $\epsilon, \delta \rightarrow 0$. We combine this result with (4.8) and (4.9) and derive that

$$\begin{aligned} \frac{d}{ds} \mu(\gamma(s)) &\geq -K(s)\mu(\mathbf{x}^* - \mathbf{z}^*) - K(s)\mu(\chi(s) - \psi(s)) \\ &\quad + I_1(\epsilon, \delta) + J_1(\epsilon, \delta) + J_3(\epsilon, \delta) - 4\epsilon. \end{aligned}$$

We let $\epsilon, \delta \rightarrow 0$ and use that $\mathbf{x}^* - \mathbf{z}^* \rightarrow \chi(s) - \psi(s)$, and $I_1(\epsilon, \delta) + J_1(\epsilon, \delta) + J_3(\epsilon, \delta) - 4\epsilon \rightarrow 0$ to obtain the desired result (4.2). ■

Remark 4.1. Theorem 4.1 holds if we require (*) to hold for almost all (t, \mathbf{x}) and (t, \mathbf{y}) in G .

5. EXISTENCE THEOREM AND STABILITY RESULTS FOR THE SOLUTIONS TO LINEAR TRANSPORT EQUATIONS WITH μ -MONOTONE COEFFICIENTS

In this section we prove existence of a Lipschitz solution to problem (3.1)–(3.2) with a p -monotone transport velocity A . We show its stability under any mollification of the coefficients A_i , C , and D in time and space. The technique is based on the ideas of Conway, developed in [4].

We also state similar results for existence of a continuous solution to the homogeneous problem (3.1)–(3.2) and its stability under the particular mollification

$$A_i^n = (-n \vee A_i \wedge n) * \xi^{1/n}, \quad i = 1, \dots, d, \tag{S}$$

where we assume weaker conditions on A . For the theory developed in this case, we refer the reader to [11]. After we combine Theorem 3.3.3 from [11] and Theorem 4.1, we get the following theorem.

THEOREM 5.1. *Let A satisfy the conditions of Theorem 4.1 and the distributional derivative $D_i A_i$ of A_i with respect to $x_i, i = 1, \dots, d$, is a bounded measure on $[0, T] \times K$ for all compact sets $K \subset \mathbb{R}^d$. Then for any $\varphi_0 \in C(\mathbb{R}^d)$, there exists a continuous weak solution to the homogeneous problem (3.1)–(3.2) of the form $\varphi(t, \mathbf{x}) = \varphi_0(\chi(0; (t, \mathbf{x})))$. Moreover, this solution is the only weak solution to (3.1)–(3.2) which is stable under the smoothing (S) of the transport velocity A and the initial condition φ_0 .*

Recall that $\varphi \in C([0, T] \times \mathbb{R}^d)$ is a weak solution to the homogeneous problem (3.1)–(3.2) with $A \in (L^1_{loc}([0, T] \times \mathbb{R}^d))^d$ if

$$\begin{aligned} & \int_0^T \int_{\mathbb{R}^d} \varphi \Phi_t + \varphi \operatorname{div}_{\mathbf{x}}(A\Phi) \, d\mathbf{x} \, dt \\ &= \int_{\mathbb{R}^d} \varphi(T, \mathbf{x})\Phi(T, \mathbf{x}) \, d\mathbf{x} - \int_{\mathbb{R}^d} \varphi_0(0, \mathbf{x})\Phi(0, \mathbf{x}) \, d\mathbf{x}, \end{aligned}$$

for all test functions $\Phi \in C^\infty([0, T] \times \mathbb{R}^d)$ with compact support.

From now on we consider only p -monotone velocities $A, 1 \leq p < \infty$. First we prove the following lemma.

LEMMA 5.1. *If A satisfies condition (B) and is a p -monotone vector function on $G, 1 \leq p < \infty$, then the derivative $D_i A_i$ of A_i with respect to x_i is a signed measure on G , having locally finite total variation.*

Proof. Let us define $E(t, \mathbf{x}) := A(t, \mathbf{x}) + \frac{1}{p}K(t)\mathbf{x}$. We let $y_j = x_j$ for all $j \neq i$ in (*) and get

$$|x_i - y_i|^{p-2} (x_i - y_i)[A_i(t, \mathbf{x}) - A_i(t, \mathbf{y})] \geq -\frac{K(t)}{p} |x_i - y_i|^p .$$

Hence we have

$$\frac{A_i(t, \mathbf{x}) - A_i(t, \mathbf{y})}{x_i - y_i} + \frac{K(t)}{p} \geq 0.$$

In terms of E_i , the above inequality can be expressed as

$$(x_i - y_i)[E_i(t, \mathbf{x}) - E_i(t, \mathbf{y})] \geq 0,$$

and the proof is completed as the proof of Lemma 2 in [4]. ■

From Lemma 5.1 and Theorem 5.1 we derive the next theorem.

THEOREM 5.2. *Let A satisfy the conditions of Theorem 4.1 with $\mu(\mathbf{z}) = \|\mathbf{z}\|_p^p$, where $1 \leq p < \infty$. Then for any $\varphi_0 \in C(\mathbb{R}^d)$, there exists a continuous weak solution to the homogeneous problem (3.1)–(3.2) of the form $\varphi(t, \mathbf{x}) = \varphi_0(\chi(0; (t, \mathbf{x})))$. Moreover, this solution is the only weak solution to (3.1)–(3.2) which is stable under the smoothing (S) of the transport velocity A and the initial condition φ_0 .*

Now we continue with a construction of a solution φ to problem (3.1)–(3.2) as a limit of the solutions φ^h to (3.5)–(3.6). Recall that the mollification here is with respect to time and space. We shall use a compactness argument to extract a convergence subsequence from the family $\{\varphi^h\}_{h>0}$ of approximate solutions.

First we prove an auxiliary lemma. Let us define by J^h the Jacobian of A^h with respect to \mathbf{x} and with $(J^h(t, \mathbf{x}))_{i,j}$ its i, j th entry and let K^h be the convolution product of K with ζ^h . Here we define $K(t) = 0$ for $t \notin [0, T]$. Also, we denote

$$\mu_p(z) := \begin{cases} |z|^{p-2} z, & \text{if } z \neq 0, \\ 0, & \text{if } z = 0. \end{cases}$$

Then the following lemma holds.

LEMMA 5.2. *If A is a p -monotone vector function on G , $1 \leq p < \infty$, then for every vector $\mathbf{z} \in \mathbb{R}^d$ and for almost all $t \in (0, T)$, we have*

$$\sum_{i=1}^d \sum_{j=1}^d (J^h(t, \mathbf{x}))_{i,j} \mu_p(z_i) z_j \geq -\frac{K^h(t)}{p} \|\mathbf{z}\|_p^p, \quad \text{for all } \mathbf{x} \in \mathbb{R}^d. \quad (5.1)$$

Proof. Since A is p -monotone and the kernel ζ^h is positive, we have

$$\sum_{i=1}^d \mu_p(x_i - y_i) (A_i^h(t, \mathbf{x}) - A_i^h(t, \mathbf{y})) \geq -\frac{K^h(t)}{p} \|\mathbf{x} - \mathbf{y}\|_p^p. \quad (5.2)$$

We choose \mathbf{z} to be an arbitrary element in \mathbb{R}^d and let $x_i = y_i + z_i s$, where $s > 0$. Then $\mu_p(x_i - y_i) = \mu_p(z_i s) = s^{p-1} \mu_p(z_i)$ and from (5.2) we obtain

$$s^{p-1} \sum_{i=1}^d \mu_p(z_i) (A_i^h(t, \mathbf{y} + \mathbf{z}s) - A_i^h(t, \mathbf{y})) \geq -\frac{K^h(t)}{p} s^p \|\mathbf{z}\|_p^p.$$

If we fix t , $A^h(t, \mathbf{x})$ is a differentiable map and hence $A^h(t, \mathbf{y} + \mathbf{z}s) - A^h(t, \mathbf{y}) = J^h(t, \mathbf{y})\mathbf{z}s + o(s)$, as $s \rightarrow 0$. Therefore,

$$s^p \sum_{i=1}^d \mu_p(z_i) \sum_{j=1}^d (J^h(t, \mathbf{y}))_{i,j} z_j + s^{p-1} \sum_{i=1}^d \mu_p(z_i) o(s) \geq -\frac{K^h(t)}{p} s^p \|\mathbf{z}\|_p^p.$$

If we divide by s^p and let $s \rightarrow 0$, we get (5.1) and the proof is completed. ■

Now we are ready to prove an existence theorem for solutions to problem (3.1)–(3.2).

THEOREM 5.3. *Assume that*

- (i) C and D are locally Lipschitz continuous on G ,
- (ii) A_1, \dots, A_d are bounded on G real valued measurable functions,
- (iii) $A = (A_1, \dots, A_d)$ is p -monotone, $1 \leq p < \infty$,
- (iv) φ_0 is locally Lipschitz continuous on \mathbb{R}^d .

Then

(a) *there is a function φ , which is locally Lipschitz continuous on G , satisfying (3.1) almost everywhere and the initial condition (3.2),*

(b) *if φ_0 has compact support in \mathbb{R}^d , and C has compact support in G , then φ has compact support in G .*

Proof. Consider the solution φ^h to (3.5)–(3.6), given by (3.7). Let $z^h(s) := z^h(s; (t, \mathbf{x}))$ be any partial derivative of $\chi^h(s; (t, \mathbf{x}))$ with respect to one of the variables x_1, \dots, x_d . Since C, D , and φ_0 are locally Lipschitz continuous, the partial derivatives of φ^h will be locally bounded uniformly in h , if all \mathbf{z}^h are locally bounded uniformly in h and s .

Let us differentiate (3.8). Then

$$\frac{d\mathbf{z}^h}{ds} = J^h(t, \chi^h)\mathbf{z}^h. \tag{5.3}$$

Using Lemmas 4.1, 4.2, and 5.2, and equality (5.3), we obtain

$$\begin{aligned} \frac{1}{p} \frac{d}{ds} \|\mathbf{z}^h(s)\|_p^p &= \sum_{i=1}^d \mu_p(z_i^h(s)) \frac{dz_i^h}{ds} \\ &= \sum_{i=1}^d \mu_p(z_i^h(s)) \left[\sum_{j=1}^d (J^h(s, \chi^h))_{i,j} z_j^h(s) \right] \geq -\frac{K^h(s)}{p} \|\mathbf{z}^h(s)\|_p^p. \end{aligned}$$

Hence for all $0 \leq s \leq t$

$$\frac{d}{ds} \|\mathbf{z}^h(s)\|_p^p \geq -K^h(s) \|\mathbf{z}^h(s)\|_p^p,$$

which leads to

$$\|\mathbf{z}^h(s)\|_p^p \leq \|\mathbf{z}^h(t)\|_p^p \exp^{\int_s^t K^h(\tau) d\tau} \leq \|\mathbf{z}^h(t)\|_p^p \exp^{\int_0^t K(\tau) d\tau}.$$

The proof can be finished as in [4]. Part (b) follows directly from formula (3.7) for φ^h and the proof is completed. ■

We shall impose an additional assumption on A to guarantee that the constructed solution φ is stable under smoothing of the coefficients, i.e., that the above derivation does not depend on the particular mollifier, used in the construction.

THEOREM 5.4. *Suppose that A, D, C satisfy the conditions of Theorem 5.1 and every Full solution χ of (3.3) that passes through the point (t, \mathbf{x}) is uniquely determined on $[0, t]$. Then (3.1)–(3.2) has a generalized locally Lipschitz continuous solution*

$$\begin{aligned} \varphi(t, \mathbf{x}) = & \varphi_0(\chi(0; (t, \mathbf{x}))) \exp \int_0^t D(\tau, \chi(\tau; (t, \mathbf{x}))) d\tau \\ & + \int_0^t C(s, \chi(s; (t, \mathbf{x}))) \exp \int_s^t D(\tau, \chi(\tau; (t, \mathbf{x}))) d\tau ds, \end{aligned} \quad (5.4)$$

which is stable under smoothing of the coefficients of (3.1).

Proof. The solution of the smooth problem (3.5)–(3.6) is given by formula (3.7). From Lemma 3.2, it follows that under the assumptions on A the family φ^h , $h > 0$, has only one limit, i.e.,

$$\lim_{h \rightarrow 0} \varphi^h(t, \mathbf{x}) = \varphi(t, \mathbf{x}),$$

uniformly on compact subsets of G . If we let $h \rightarrow 0$ in (3.7), we obtain that the limit function φ is given explicitly by formula (5.4). This function is locally Lipschitz continuous and satisfies the initial condition and Eq. (3.1) almost everywhere in G . ■

COROLLARY 5.1. *Problem (3.1)–(3.2) has a locally Lipschitz continuous solution, which is stable under smoothing of the coefficients, if D and C are locally Lipschitz continuous on G and any of the following conditions is satisfied:*

- (i) A is continuous, bounded, and p -monotone, $1 \leq p < \infty$,
- (ii) $A(s, \mathbf{x}) = A(\mathbf{x})$ is bounded and p -monotone, $1 < p < \infty$,

(iii) A is bounded and for some $1 < p < \infty$ satisfies the inequality

$$\sum_{i=1}^d \mu_p(x_i - y_i)(A_i(t, \mathbf{x}) - A_i(s, \mathbf{y})) \geq -\frac{K}{p}(\|\mathbf{x} - \mathbf{y}\|_p^p + |t - s|^p), \quad (5.5)$$

for all $(t, \mathbf{x}), (s, \mathbf{y}) \in G$ with K an absolute constant.

Proof. (i) If A is continuous, then

$$M_{(s, \mathbf{x})}\{A(\cdot, \cdot)\} = M_{\mathbf{x}}\{A(s, \cdot)\}, \quad (5.6)$$

and therefore the Full solutions to (3.3)–(3.4) are Filippov and vice versa. Using the rest of the assumptions in (i), we apply Theorem 4.1 and derive that every Full solution of (3.3)–(3.4) is unique on the left. Then, Theorem 5.4 implies that our solution to (3.1)–(3.2) is stable under smoothing of the coefficients.

(ii) If $A(s, \mathbf{x}) = A(\mathbf{x})$, (5.6) holds again and the proof goes as above.

(iii) Inequality (5.5) means that $A(\bar{\mathbf{x}})$, defined in (3.11), is p -monotone. Then, by Theorem 4.1, (3.10) will have uniquely determined on the left Filippov solutions. Hence the Full solutions of (3.3) are unique on the left. But (5.5) implies also, that $A(s, \mathbf{x})$ is p -monotone. Therefore, by Theorem 5.4 our solution φ of (3.1)–(3.2) does not depend on the particular mollifier, used in the construction. ■

6. UNIQUENESS QUESTION FOR MULTIDIMENSIONAL QUASILINEAR CONSERVATION LAWS

We consider the quasilinear conservation law (3.13)–(3.14). Let u_1 and u_2 be two weak solutions of (3.13)–(3.14) and let us denote $F(t, \mathbf{x}) := (F_1(t, \mathbf{x}), \dots, F_d(t, \mathbf{x}))$, where

$$\begin{aligned} F_i(t, \mathbf{x}) &:= \frac{f_i(t, \mathbf{x}, u_1(t, \mathbf{x})) - f_i(t, \mathbf{x}, u_2(t, \mathbf{x}))}{u_1(t, \mathbf{x}) - u_2(t, \mathbf{x})} \\ &= \int_0^1 \frac{\partial f_i}{\partial u}(t, \mathbf{x}, u_1 + \tau(u_2 - u_1)) d\tau. \end{aligned}$$

Then the following uniqueness theorem holds.

THEOREM 6.1. *Let u_1 and u_2 be two weak solutions of (3.13)–(3.14). Let for some $1 \leq p < \infty$ and for almost all $t \in (0, T)$ the function $F(t, \mathbf{x})$ satisfy the inequality*

$$\sum_{i=1}^d \mu_p(x_i - y_i)(F_i(t, \mathbf{x}) - F_i(t, \mathbf{y})) \leq \frac{K(t)}{p} \|\mathbf{x} - \mathbf{y}\|_p^p, \quad (6.1)$$

for all $\mathbf{x}, \mathbf{y} \in G$, where K is a non-negative function, integrable on $[0, T]$. Then almost everywhere in G

$$u_1 = u_2.$$

Proof. The proof follows the arguments in [4]. Instead of Theorem 1 from [4] we need Theorem 5.3 to show the existence of a solution to a problem, dual to the conservation law (3.13)–(3.14). ■

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