Math 414, Homework 1

1. Let $z = a + ib$ be a complex number. Show that

   (a) $z\bar{z} = a^2 + b^2$

   (b) $a = \frac{z+\bar{z}}{2}$

   (c) $b = \frac{z-\bar{z}}{2i}$

   (d) Compute $|3 - 2i|$ and find the exponential and polar form of this number.

   (e) Show that $\overline{e^{i\theta}} = e^{-i\theta}$

   (f) Show that $e^{i(\theta+\pi)} = -e^{i\theta}$

2. (a) Verify that the function

$$< V, W >= \sum_{j=1}^{n} v_j \bar{w}_j, \quad V = (v_1, \dots, v_n), \quad W = (w_1, \dots, w_n) \in C^n$$

   defines an inner product on C^n.

   (b) Suppose u_0 and u_1 are vectors in the inner product space V and

$$< u_0, v >=< u_1, v >, \quad \text{for all } v \in V.$$

   Show that $u_0 = u_1$.

3. (a) Show that a set of orthonormal vectors is linearly independent.

   (b) Show that the functions

$$\{\frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}} \sin(nt), \frac{1}{\sqrt{\pi}} \cos(nt)\}, \quad n = 1, 2, \dots$$

   are elements of the space $L^2[-\pi, \pi]$ and are orthonormal set of functions.

   (c) Let $V_N := \text{span}\{\frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}} \sin(nt), \frac{1}{\sqrt{\pi}} \cos(nt), n = 1, 2, \dots, N\}$. Find the orthogonal projection f_N of $f(t) = t^2$ onto V_N.

   (d) Plot f, f_2, f_4, f_8 on the interval $[-\pi, \pi]$ (use a program to compute the orthogonal projections)

4. Consider the space $V_1 = \text{span}\{\Phi, H\}$, where $\Phi(t) = \chi_{[0,1)}(t)$ is the characteristic function of $[0, 1)$, and H is the Haar scaling function, i.e. $H(t) = 1$ for $t \in [0, 1/2)$, $H(t) = -1$ for $t \in [1/2, 1)$, and $H(t) = 0$ outside the interval $[0, 1)$.

   (a) Show that Φ and H are orthonormal in $L_2([0, 1])$

   (b) Find the orthogonal projection of $f(t) = t^2$ onto V_1.