Math 414, Homework 2

0. Gram-Schmidt orthogonalization procedure: describes a way to construct an orthonormal basis.

Let $V_0 = \text{span}\{v_1, \ldots, v_n\}$, where the basis $v_1, \ldots, v_n$ is not necessarily orthonormal. We construct an orthonormal basis $e_1, \ldots, e_n$ the following way:

Set $e_1 = v_1/||v_1||$. Clearly $||e_1|| = 1$. Assume that we have already constructed $e_1, \ldots, e_{k-1}$ that are orthonormal. Consider the vector $v_k$ and its orthogonal projection $v_k^0$ on the space $\text{span}\{e_1, \ldots, e_{k-1}\}$. From the theorems in class it follows that

$$v_k^0 = <v_k, e_1> e_1 + \ldots + <v_k, e_{k-1}> e_{k-1},$$

and $(v_k - v_k^0) \perp \text{span}\{e_1, \ldots, e_{k-1}\}$. Thus, if we denote by $E_k := v_k - v_k^0$, we will have that $E_k \perp e_1, \ldots, e_{k-1}$. Then if we set $e_k = E_k/||E_k||$, we will have a vector with norm 1, orthogonal to all previous vectors $e_1, \ldots, e_{k-1}$. Therefore the system $e_1, \ldots, e_k$ will be a system of orthonormal vectors. We do this for $k = 2, 3, \ldots, n$ and end up with an orthonormal basis.

Following the above described process and the fact that $\{1, x, x^2\}$ forms a basis for the set $\pi_2$ of all polynomials of degree 2, find an orthonormal basis for $\pi_2$ with respect to the scalar product in $L^2([-1,1])$.

1. (a) Expand the function $f(t) = |\sin t|$ in a Fourier series on $[-\pi, \pi]$. Plot $f$ and the partial sums $S_5, S_{10}, S_{20}$ and $S_{40}$.

(b) Expand the function

$$f(t) := \begin{cases} 1, & -1/2 < t \leq 1/2, \\ 0, & -1 < t \leq -1/2 \text{ and } 1/2 < t \leq 1, \end{cases}$$ (1)

in a Fourier series valid on $[-1,1]$. Plot $f$ and the partial sums $S_5, S_{10}, S_{20}$ and $S_{40}$.

2. (a) Expand $f(t) = \cos t$ in a Fourier sine series on the interval $[0, \pi]$. What is the Fourier cosine series of the same function on the interval $[0, \pi]$?

(b) Show that

$$\left\{ \frac{1}{\sqrt{2a}}, \frac{1}{\sqrt{a}} \cos \left(\frac{n\pi}{a} t\right), \frac{1}{\sqrt{a}} \sin \left(\frac{n\pi}{a} t\right) \right\}, n = 1, 2, \ldots$$

is an orthonormal system of functions on $L^2([-a,a])$.

3. Gibbs phenomenon

Consider the function

$$f(t) := \begin{cases} \pi - t, & 0 \leq t \leq \pi, \\ -\pi - t, & -\pi \leq t < 0. \end{cases}$$ (2)

(a) Find the Fourier series $F(f, t)$ for $f$.

(b) Plot $f$, $S_{10}$, $S_{25}$ and $S_{50}$ on $[-\pi, \pi]$, where $S_N$ is the $N$-th partial sum of $F(f, t)$. 

(c) Denote by $E_N(t) := S_N(t) - f(t)$ the error we make if we use $S_N$ instead of $f$. Consider this function on $[0, \pi]$. Show that

$$\frac{d}{dt} E_N(t) = \frac{\sin((N + 1/2)t)}{\sin(t/2)}$$

(d) Show that $\theta_N = \frac{2\pi}{2N+1}$ is the first critical point of $E_N$ to the right of $t = 0$.

(e) Using the fundamental theorem of calculus and parts (c) and (d) show that

$$E_N(\theta_N) = \int_0^{\theta_N} \frac{\sin((N + 1/2)t)}{\sin(t/2)} \, dt - \pi$$

(f) Show that

$$\lim_{N \to \infty} E_N(\theta_N) = 2 \int_0^\pi \frac{\sin t}{t} \, dt - \pi.$$  

(use change of variables $u = (N + 1/2)t$)

(g) Show that

$$2 \int_0^\pi \frac{\sin t}{t} \, dt - \pi \approx 0.562,$$

by evaluating this integral numerically.

**Conclusion:** The amount that the $N$-th partial sum overshoots $f$ is about 0.562 when $N$ is large (Gibbs effect).

4. **Compression:** One way to compress a signal is first to express it as a Fourier series, then discard all the small Fourier coefficients and retain (transmit) only the finite number of coefficients that are larger than some given threshold.

Consider the function

$$g(t) = e^{-t^2/8} (\cos(2t) + 2\sin(4t) + 0.4\cos(2t)\cos(10t))$$

(a) Compute the partial Fourier series $S_N = a_0 + \sum_{n=1}^N (a_n \cos(nt) + b_n \sin(nt))$ for $N = 25$.

(b) Throw away any coefficients that are smaller than $\epsilon = 0.01$ in absolute value. Plot the resulting series and compare with the original function $g(t)$.

(c) Try again for $\epsilon = 0.1$ and $\epsilon = 0.001$. 