Math 417, Homework 2

1. (a) Consider the matrix
$$A = \begin{pmatrix} 1 & 2 & 5 \\ -1 & 2 & -4 \\ -8 & -1 & 2 \end{pmatrix}.$$
Compute $\|A\|_\infty$ and find a vector x such that $\|A\|_\infty = \|Ax\|_\infty / \|x\|_\infty$

(b) Let B be a $n \times n$ matrix. Show that B and $\frac{1}{2}(B + B^t)$ generate the same quadratic form.

2. Consider
$$A = \begin{pmatrix} 1.2969 & 0.8648 \\ 0.2161 & 0.1441 \end{pmatrix}, \quad b = \begin{pmatrix} 0.8642 \\ 0.1440 \end{pmatrix}, x = \begin{pmatrix} 2 \\ -2 \end{pmatrix}, x_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, x_2 = \begin{pmatrix} 0.9911 \\ -0.4870 \end{pmatrix}.$$
(a) Show that x is the exact solution of $AX = b$.

(b) compute the errors e_1, e_2 and residuals r_1 and r_2 for the approximate solutions x_1 and x_2.

3. Show that

(a) $\kappa(A) \geq 1$ for any matrix A.

(b) $\kappa(AB) \leq \kappa(A)\kappa(B)$

4. (a) Show that the expression $\|x\|_2 = \sqrt{\sum_{i=1}^n |x_i|^2}$ defines a vector norm

(b) Let S be a real and nonsingular matrix, and let $\|\cdot\|$ be any norm on R^n. Define $\|\cdot\|'$ by $\|x\|' = \|Sx\|$. Show that $\|\cdot\|'$ is also a norm on R^n.

(c) Prove that the $\|\cdot\|_1$ matrix norm can be computed by
$$\|A\|_1 := \max_{x \neq 0} \frac{\|Ax\|_1}{\|x\|_1} = \max_j \sum_{i=1}^n |a_{ij}|.$$

5. Consider
$$A = \begin{pmatrix} 4 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 4 \end{pmatrix}, \quad b = \begin{pmatrix} 5 \\ 6 \\ 5 \end{pmatrix}$$
(a) Find $\|B\|_\infty$, $\rho(B)$, where B is the iteration matrix for the Jacobi and Gauss-Seidel methods.

(b) Use the theorem given in class to find the optimal SOR parameter (you should check that A satisfies the hypothesis of the theorem).

(c) Perform 10 iterations of Jacobi, Gauss-Seidel, and optimal SOR using $x^0 = 0$. Make a table containing the following information:

column 1: k (iteration step number)

column 2: x_1^k (first component of solution vector)

column 3: x_2^k (second component of solution vector)

column 4: $\|e_k\|_\infty$ (norm of error at step k)

column 5: $\|e_k\|_\infty / \|e_{k-1}\|_\infty$ (ratio of successive error norms).

Discuss the results.