Math 414, Homework 3

1. (a) Let
\[ f(t) := \begin{cases} 
\sin(3t), & -\pi \leq t \leq \pi, \\
0, & \text{otherwise}. 
\end{cases} \]  
(1)

Compute the Fourier transform \( \mathcal{F}[f](\lambda) \)

(b) Let
\[ f(t) := \begin{cases} 
t^2 + 4t + 4, & -2 \leq t \leq -1, \\
2 - t^2, & -1 \leq t \leq 1, \\
t^2 - 4t + 4, & 1 \leq t \leq 2, \\
0, & \text{otherwise}. 
\end{cases} \]  
(2)

Show that \( f \) and \( f' \) are continuous everywhere (see what happens at \( t = -2, t = -1, t = 1 \) and \( t = 2 \)). Plot the graph of \( f \). Compute the Fourier transform \( \mathcal{F}[f](\lambda) \)

Note that in (a), \( f \) is continuous, but not differentiable and \( \mathcal{F}[f](\lambda) \) decays like \( \frac{1}{\sqrt{\lambda}} \), while in (b) \( f \) and \( f' \) are continuous and \( \mathcal{F}[f](\lambda) \) decays like \( \frac{1}{\lambda^3} \).

2. (a) Let
\[ \psi(t) := \begin{cases} 
1, & 0 \leq t < 1, \\
0, & \text{otherwise}. 
\end{cases} \]  
(3)

Compute the convolution \( \psi * \psi \).

(b) Let \( f \in L^2(-\infty, \infty) \) be a real-valued even function. Show that its Fourier transform \( \mathcal{F}[f](\lambda) \) is real-valued. What is happening when \( f \) is a real-valued odd function?

3. Oversampling

(a) Suppose \( f \) is a band-limited signal with \( \mathcal{F}[f](\lambda) = 0 \) for \( |\lambda| \geq \Omega \). Fix a number \( a \) > 1. Repeat the proof of the Shannon-Whittaker Sampling theorem to show that
\[ \mathcal{F}[f](\lambda) = \sum_{n=-\infty}^{\infty} c_n e^{-in\pi\lambda/a\Omega}, \quad c_n = \frac{\pi}{\sqrt{2\pi a\Omega}} f \left( \frac{n\pi}{a\Omega} \right). \]

(b) Let \( g_a(t) \) is such that its Fourier transform \( \mathcal{F}[g_a](\lambda) \) is given by
\[ \mathcal{F}[g_a](\lambda) = \begin{cases} 
0, & |\lambda| \geq a\Omega, \\
\frac{\lambda + a\Omega}{(a-1)\Omega}, & -a\Omega \leq \lambda \leq -\Omega, \\
1, & |\lambda| \leq \Omega, \\
\frac{\lambda - a\Omega}{(1-a)\Omega}, & \Omega \leq \lambda \leq a\Omega. 
\end{cases} \]  
(4)

Find \( g_a(t) \).

(c) Since \( \mathcal{F}[f](\lambda) = 0 \) for \( |\lambda| \geq \Omega \), the Fourier Transform of \( f \) can be represented as \( \mathcal{F}[f](\lambda) = \mathcal{F}[f](\lambda) \mathcal{F}[g_a](\lambda) \). Use the properties of the Fourier Transform and (a) and (b) to show that
\[ f(t) = \sum_{n=-\infty}^{\infty} \frac{\pi}{\sqrt{2\pi a\Omega}} f \left( \frac{n\pi}{a\Omega} \right) g_a \left( t - \frac{n\pi}{a\Omega} \right). \]
Since $g_a(t)$ has a factor of $t^2$ in the denominator, this expression for $f(t)$ converges faster than the expression for $f$ given in the Shannon-Whittaker Sampling theorem. The disadvantage of the formula in (c) is that the function is sampled on a grid $n \pi/(a \Omega)$ which is more frequent rate of sampling than the grid $n \pi/\Omega$ used in the theorem.

4. Consider the signal

$$g(t) = e^{-t} (\sin(5t) + \sin(3t) + \sin(t) + \sin(40t)), \quad 0 \leq t \leq \pi.$$ 

Filter this signal with the Butterworth filter (i.e. compute $(f \ast h)(t)$ for $0 \leq t \leq \pi$). Try various values of $A = a$ (starting with $A = a = 10$). Compare (plot) the filtered signal with the original signal.