Math 417, Homework 3

1. (a) A monomial matrix is a square matrix in which each row and column contains exactly one nonzero entry. Prove that a monomial matrix is nonsingular.

   (b) A square matrix A is said to be skew-symmetric if $A^T = -A$. Prove that if A is skew-symmetric, then $x^T A x = 0$ for all x.

2. Consider the $N \times N$ tridiagonal matrix

$$A = \frac{1}{h^2} \begin{pmatrix} 2\epsilon + h^2 & -\epsilon & 0 & \dots & 0 & 0 & 0 \\ -\epsilon & 2\epsilon + h^2 & -\epsilon & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & -\epsilon & 2\epsilon + h^2 & -\epsilon \\ 0 & 0 & 0 & \dots & 0 & -\epsilon & 2\epsilon + h^2 \end{pmatrix}$$

   and the vector $f = (f_1, f_2, \dots, f_N)^T$, where $f_i = 2ih + 1$, $h = 2^{-n}$, $N = 2^n - 1$, $n = 1, 2, \dots, 8$. Take $\epsilon = 10^{-3}$ and solve the system $Au = f$ using LU factorization for a tridiagonal matrix as discussed in class. In your code do not create the matrix A. For each value of $n = 1, 2, \dots, 8$, plot the points (u_i, x_i), $i = 1, \dots, N$, where u is the solution of the system and $x_i = ih$.

3. Given the block tridiagonal matrix

$$A = \begin{pmatrix} B & -I & 0 \\ -I & B & -I \\ 0 & -I & B \end{pmatrix}, \quad \text{where} \quad B = \begin{pmatrix} 4 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 4 \end{pmatrix},$$

   solve the linear system $Ax = b$, with $b^T = (0, 0, 1, 0, 0, 1, 0, 0, 1)$ using the Jacobi and Gauss-Seidel methods with starting guess $x^{(0)} = 0$. Terminate the iteration when the maximum residual is less than 10^{-2}. To avoid an infinite loop, set the maximum number of iterations to 100. For each method (J, GS) make a table which displays the following information:

   column 2: k (number of iterations needed)

   column 3: $\|r^{(k)}\|_\infty$ (norm of residual at final step)

   column 4: $\|r^{(k)}\|_\infty / \|r^{(k-1)}\|_\infty$ (ratio of successive residual norms at final step)

   Summarize the results. Are the results consistent with the theorems discussed in class?

4. (a) Show that $D_+ D_+ f = f''(x) + O(h)$

   (b) Show that $D_+ D_- f = D_- D_+ f$