Math 417, Homework 3

1. (a) A monomial matrix is a square matrix in which each row and column contains exactly one nonzero entry. Prove that a monomial matrix is nonsingular.

(b) A square matrix \( A \) is said to be skew-symmetric if \( A^T = -A \). Prove that if \( A \) is skew-symmetric, then \( x^T Ax = 0 \) for all \( x \).

2. Consider the \( N \times N \) tridiagonal matrix

\[
A = \frac{1}{h^2} \begin{pmatrix}
2\epsilon + h^2 & -\epsilon & 0 & \ldots & 0 & 0 & 0 \\
-\epsilon & 2\epsilon + h^2 & -\epsilon & \ldots & 0 & 0 & 0 \\
0 & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
0 & 0 & 0 & \ldots & -\epsilon & 2\epsilon + h^2 & -\epsilon \\
0 & 0 & 0 & \ldots & 0 & -\epsilon & 2\epsilon + h^2
\end{pmatrix}
\]

and the vector \( f = (f_1, f_2, \ldots, f_N)^T \), where \( f_i = 2ih + 1 \), \( h = 2^{-n} \), \( N = 2^n - 1 \), \( n = 1, 2, \ldots, 8 \). Take \( \epsilon = 10^{-3} \) and solve the system \( Au = f \) using \( LU \) factorization for a tridiagonal matrix as discussed in class. In your code do not create the matrix \( A \). For each value of \( n = 1, 2, \ldots, 8 \), plot the points \((u_i, x_i)\), \( i = 1, \ldots, N \), where \( u \) is the solution of the system and \( x_i = ih \).

3. Given the block tridiagonal matrix

\[
A = \begin{pmatrix}
B & -I & 0 \\
-I & B & -I \\
0 & -I & B
\end{pmatrix}, \quad \text{where} \quad B = \begin{pmatrix}
4 & -1 & 0 \\
-1 & 4 & -1 \\
0 & -1 & 4
\end{pmatrix},
\]

solve the linear system \( Ax = b \), with \( b^T = (0, 0, 1, 0, 0, 1, 0, 0, 1) \) using the Jacobi and Gauss-Seidel methods with starting guess \( x^{(0)} = 0 \). Terminate the iteration when the maximum residual is less than \( 10^{-2} \). To avoid an infinite loop, set the maximum number of iterations to 100. For each method (J, GS) make a table which displays the following information:

- column 2: \( k \) (number of iterations needed)
- column 3: \( \|r^{(k)}\|_\infty \) (norm of residual at final step)
- column 4: \( \|r^{(k)}\|_\infty /\|r^{(k-1)}\|_\infty \) (ratio of successive residual norms at final step)

Summarize the results. Are the results consistent with the theorems discussed in class?

4. (a) Show that \( D_\perp D_\perp f = f''(x) + O(h) \)

(b) Show that \( D_\perp D_- f = D_- D_\perp f \)