Math 414, Homework 4

1. **The Z transform**

   The Z-transform of a sequence $x = (\ldots, x_{-1}, x_0, x_1, \ldots) \in \ell^2$ is the function $\hat{x} : [-\pi, \pi] \to C$, defined by

   $$\hat{x}(\phi) = \sum_{j=-\infty}^{\infty} x_j e^{-ij\phi}$$

   (a) **The Z transform is a generalization of the discrete Fourier transform**

   Show that

   $$\hat{x}\left(\frac{2\pi k}{n}\right) = \left(\mathcal{F}_n[x]\right)_k ,$$

   where $x = (x_0, \ldots, x_{n-1})$ is a finite sequence.

   (b) Find the Z-transform of the sequence $x = (\ldots, 0, 0, 1, \frac{1}{2}, \frac{1}{4}, \ldots, \frac{1}{2^n}, \ldots)$.

   (c) **Connection with Fourier series**

   Let $f \in L^2[-\pi, \pi]$ with Fourier series $\mathcal{F}f$. Show that

   $$\hat{x}(\phi) = \mathcal{F}f(-\phi),$$

   where $x = (\ldots, x_{-1}, x_0, x_1, \ldots, x_n, \ldots)$, and x_n is the n-th Fourier coefficient of f.

   (d) **Isometry (preservation of inner product up to a constant)**

   If $\{x_n\}$ and $\{y_n\}$ are the Fourier coefficients of f and $g \in L^2[-\pi, \pi]$,respectively show that

   $$\frac{1}{2\pi} < \hat{x}, \hat{y} >_{L^2[-\pi,\pi]} = < x, y >_{\ell^2} .$$

2. **Filtering**

   (a) Consider the signal, generated by the function

   $$y(t) = e^{-(\cos t)^2}(\sin(2t) + 2\cos(4t) + 0.4 \sin t \sin(50t)), \quad t \in [0, 2\pi].$$

   The signal is discretized by evaluating it at $2^8 = 256$ equally spaced points on $[0, 2\pi]$. Then the fast Fourier transform is used to generate the discrete Fourier coefficients $\hat{y}_k$, $k = 0, \ldots, 255$. Keep only $\hat{y}_k$, $k = 0, \ldots, 5$ and set $\hat{y}_k = 0$ for $k = 6, \ldots, 128$. Since $\hat{y}_{n-k} = \overline{\hat{y}_k}$ for real sequences, we will have $\hat{y}_k = 0$ for $k = 128, \ldots, 250$. Apply the inverse fast Fourier transform to the filtered $\hat{y}_k$, assemble the filtered signal and graph it, together with the original noisy signal g.

   (b) Let

   $$f(t) = e^{-t^2/10}(\sin(2t) + 2\cos(4t) + 0.4 \sin t \sin(50t)), \quad t \in [0, 2\pi].$$

   Discretize f by setting $y_k = f(\frac{2k\pi}{256})$, $k = 1, \ldots, 256$. Use the fast Fourier transform to compute $\hat{y}_k$. Recall that $\hat{y}_{n-k} = \overline{\hat{y}_k}$, and therefore the low frequency coefficients are $\hat{y}_0, \ldots, \hat{y}_m$ and $\hat{y}_{256-m}, \ldots, \hat{y}_{256}$ for some value of m. Filter out the high frequency terms by setting $\hat{y}_k = 0$, $k = m, \ldots, 255 - m$ with $m = 6$, then apply the fast Fourier transform to this new set of $\hat{y}_k$ to compute the y_k (now filtered), plot the new values of y_k and compare with the original function. Experiment with other values of m.

3. **Compression**

   (a) Let

$$g(t) = e^{-t^2/10}(\sin(2t) + 2\cos(4t) + 0.4\sin t\sin(10t)), \quad t \in [0, 2\pi].$$

   We wish to compress this signal by taking its fast Fourier transform and ignoring the small Fourier coefficients. We sample the signal at $2^8 = 256$ equally spaced nodes. We apply the fast Fourier transform to generate $\hat{y}_k$ and set 80% of the y_k (the smallest 80%) equal to zero. Taking the inverse fast Fourier transform of the new $\hat{y}_k$, we assemble the compressed signal y_c. Plot y and y_c. Calculate the relative error between the original signal y and the compressed signal y_c, that is

$$E = \frac{\|y - y_c\|_{\ell^2}}{\|y\|_{\ell^2}}$$

   (b) Let $tol = 1.0$ and consider the function from [2b]. If $|\hat{y}_k| < tol$, then set $\hat{y}_k = 0$. Apply the inverse fast Fourier transform to this new set of $\hat{y}_k$ to compute the y_k. Plot the new values of y_k and compare with the original function. Experiment with other values of tol. Keep track of percentage of Fourier coefficients that have been filtered out. Compute the relative error between the original signal y and the compressed signal.

4. Define the time translation operator T_p on a sequence x by

$$[T_p(x)]_k = x_{k-p},$$

   i.e. T_p takes a sequence x and shifts it p units to the right. Show that $T_p(e^n) = e^{n+p}$, where e^n is the sequence $e^n = (\ldots, 0, 0, 1, 0, 0 \ldots)$, with 1 on n-th position.