Math 417, Homework 4

1. Establish the formula:

$$f''(x) \approx \frac{2}{h^2}\left(\frac{f(x_0)}{(1+\alpha)} - \frac{f(x_1)}{\alpha} + \frac{f(x_2)}{\alpha(1+\alpha)}\right),$$

using the unevenly spaced points $x_0 < x_1 < x_2$, where $x_1 - x_0 = h$ and $x_2 - x_1 = \alpha h$. Notice that this formula reduces to the standard central-difference formula when $\alpha = 1$.

2. (a) Derive the following rule for estimating f'''

$$f'''(x) \approx \frac{1}{2h^3}\left(f(x+2h) - 2f(x+h) + 2f(x-h) - f(x-2h)\right),$$

and the corresponding error term.

(b) Use the centered difference approximation $D_0 f$ to estimate $f'(x)$ for $f(x) = \frac{1}{1+x}$ at $x = 0$ with $h = 0.01$. Find the absolute error.

3. Let $f(x) = \frac{1}{1+x^2}$ and $g(x) = \cos x$. Compute $f'(0)$ and $g'(0)$ using $D_+ f := (f(x+h) - f(x))/h$ and $D_0 f := (f(x+h) - f(x-h))/2h$ and step-size $h = 2^{-n}$, $n = 1, 2, \ldots, 5$. For each difference formula ($D_+ f$ and $D_0 f$) and each function (f and g), make a table which contains the following information

column 1: h

column 2: Df

column 3: $f'(0) - Df$

column 4: $(f'(0) - Df)/h$

column 5: $(f'(0) - Df)/h^2$

where D is being $D_+ f$ and $D_0 f$. Explain what is happening in each column. (at the end you need to present 4 different tables)

4. Use the finite difference scheme to solve the 2-point boundary value problem

$$-\epsilon\phi'' + \phi = 2x + 1, \quad 0 < x < 1, \quad \phi(0) = \phi(1) = 0.$$

Use mesh size $h = 2^{-n}$, $n = 1, \ldots, 8$. Take $\epsilon = 10^{-3}$. Solve $A_h u = f$ using LU factorization for a triagonal system as discussed in class. In your code do not create the matrix A_h. Make a table which contains the following information

For each difference formula, make a table which contains the following information

column 1: h

column 2: $\max_i |\phi_i - u_i|$

column 3: $\max_i |\phi_i - u_i|/h$

column 4: $\max_i |\phi_i - u_i|/h^2$

column 5: $\max_i |\phi_i - u_i|/h^3$

Discuss the results. What order of accuracy is attained?