Math 609, Homework 4

1. (a) Show that if there is a polynomial $p$ without constant term such that
   \[ \left\| I - p(A) \right\| < 1 \]
   then $A$ is invertible. Find a formula for $A^{-1}$.

   (b) Prove that if $p$ is a polynomial with constant term $c_0$ and if
   \[ |c_0| + \left\| I - p(A) \right\| < 1, \]
   then $A$ is invertible. Find a formula for $A^{-1}$.

   (c) Show that if $\left\| AB - I \right\| = \epsilon < 1$, then
   \[ \left\| A^{-1} - B \right\| \leq \frac{\epsilon}{1 - \epsilon}. \]

2. (a) Show that the eigenvalues of a Hermitian matrix are real.

   (b) Prove that if $A$ is positive definite, then its eigenvalues are positive.

   (c) Prove that if $A$ is nonsingular and if $|\lambda| < \left\| A^{-1} \right\|^{-1}$, then $\lambda$ is not an eigenvalue of $A$.

3. (a) Show that the basic iteration process given by
   \[ Qx^{(k)} = (Q - A)x^{(k-1)} + b \]
   is equivalent to the following:
   Given $x^{(k)}$, compute $r^{(k)} = b - Ax^{(k)}$, solve for $z^{(k)}$ in the equation $Qz^{(k)} = r^{(k)}$, and define $x^{(k+1)} = x^{(k)} + z^{(k)}$.

   (b) Using the notation in (a), show that
   \[ r^{(k+1)} = (I - AQ^{-1})r^{(k)}, \quad z^{(k+1)} = (I - Q^{-1}A)z^{(k)} \]

4. (a) Program and test the conjugate gradient method in the case $A = (a_{ij})$, $a_{ij} = \frac{1}{i+j+1}$ (such a matrix is called a Hilbert matrix) and $b = (b_i)$, $b_i = \frac{1}{3} \sum_{j=1}^{n} a_{ij}$, where $A$ is $16 \times 16$ and $32 \times 32$ matrix and $x^{(0)} = 0$.

   (b) Solve the system $Ax = b$, where
   \[
   A = \begin{pmatrix}
   10 & 1 & 2 & 3 & 4 \\
   1 & 9 & -1 & 2 & -3 \\
   2 & -1 & 7 & 3 & -5 \\
   3 & 2 & 3 & 12 & -1 \\
   4 & -3 & -5 & -1 & 15 \\
   \end{pmatrix}
   \]

   $b^T = (12, -27, 14, -17, 12)$, using the 1) Jacobi, 2) Gauss-Seidel, 3) Conjugate gradient method, starting with $x^{(0)} = 0$. 
5. Given the block tridiagonal matrix

\[ A = \begin{pmatrix} B & -I & 0 \\ -I & B & -I \\ 0 & -I & B \end{pmatrix}, \quad \text{where} \quad B = \begin{pmatrix} 4 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 4 \end{pmatrix}, \]

solve the linear system \( Ax = b \), with \( b^T = (0, 0, 1, 0, 0, 0, 1, 0, 0, 1) \) using the Jacobi and Gauss-Seidel methods with starting guess \( x^{(0)} = 0 \). Terminate the iteration when the maximum residual is less than \( 10^{-2} \). To avoid an infinite loop, set the maximum number of iterations to 100. For each method (J, GS) make a table which displays the following information:

- column 2: \( k \) (number of iterations needed)
- column 3: \( \| r^{(k)} \|_\infty \) (norm of residual at final step)
- column 4: \( \frac{\| r^{(k)} \|_\infty}{\| r^{(k-1)} \|_\infty} \) (ratio of successive residual norms at final step)

Summarize the results. Are the results consistent with the theorems discussed in class?