Math 414, Homework 5

1. Let ϕ and ψ be the Haar scaling and wavelet functions respectively. Let V_j and W_j be the spaces generated by $\phi(2^j x - k)$, $k \in Z$ and $\psi(2^j x - k)$, $k \in Z$, respectively.

   (a) Consider the function defined on $[0, 1)$, given by

   $$f(t) := \begin{cases} -1, & 0 \le t < 1/4, \\ 4, & 1/4 \le t < 1/2, \\ 2, & 1/2 \le t < 3/4, \\ -3, & 3/4 \le t < 1. \end{cases} \tag{1}$$

   Express f first in terms of the basis for V_2 and then decompose f into its component parts in W_1, W_0 and V_0, i.e. find the Haar wavelet decomposition for f.

   (b) Do the same for

   $$f(t) := \begin{cases} 2, & 0 \le t < 1/4, \\ -3, & 1/4 \le t < 1/2, \\ 1, & 1/2 \le t < 3/4, \\ 3, & 3/4 \le t < 1. \end{cases} \tag{2}$$

2. (a) Reconstruct $g \in V_3$ given these coefficients in its wavelet Haar decomposition:

   $$a^2 = [1/2, 2, 5/2, -3/2], \quad b^2 = [-3/2, -1, 1/2, -1/2]$$

   The first entry in each list corresponds to $k = 0$. Plot the graph of g.

   (b) Do the same for

   $$a^1 = [3/2, -1], \quad b^1 = [-1, -3/2], \quad b^2 = [-3/2, -3/2, -1/2, -1/2]$$

3. Let

   $$f(t) = e^{-t^2/10}(\sin(2t) + 2\cos(4t) + 0.4\sin(t)\sin(50t))$$

   (a) Discretize f over the interval $[0, 1)$ using $n = 8$ as the top level (i.e. there are 2^8 nodes in the discretization). Implement the decomposition algorithm using the Haar wavelets. Plot $f_{j-1} \in V_{j-1}$ for $j = 8, \ldots, 1$ and compare with the original signal.

   (b) Filter the wavelet coefficients computed in (a) by setting to zero any wavelet coefficients whose absolute value is less than $tol = 0.1$. Then reconstruct the signal. Plot the reconstructed f_8 and compare with the original signal. Compute the l^2 difference between the original signal and the compressed signal. Experiment with various tolerances. Keep track of the percentage of wavelet coefficients that have been filtered out.

4. Haar wavelets can be used to detect a discontinuity in a signal. Let g be defined on $[0, 1)$ via

   $$g(t) := \begin{cases} 0, & 0 \le t < 7/17, \\ 1 - t^2, & 7/17 \le t < 1. \end{cases} \tag{3}$$

   Discretize g over $[0, 1)$ using $n = 7$ as the top level (i.e. there are 2^7 nodes in the discretization). Implement a one level decomposition and plot the magnitudes of the level 6 wavelet coefficients. Which wavelet has the largest coefficient? What t corresponds to this wavelet? Try the method again with $7/17$ replaced by $8/9$ and then by $2/7$. Why do you think the method works?