Math 417, Homework 6

1. Consider $f(x) = x^3 - 3$.

   (a) Show that $f(x)$ has a root α in the interval $[1, 2]$.

   (b) Compute an approximation to the root by taking 5 steps of the bisection method.

   (c) Repeat, using fixed point iteration with $g_1(x) = x - f(x)/3$ and $g_2(x) = 3/x^2$. Take $x_0 = 1.5$ for the starting value.

   (d) Repeat, using Newton's method. Take $x_0 = 1.5$ for the starting value.

   For each method, present the results in a form of a table:

   column 1: n (step)

   column 2: x_n (approximation)

   column 3: $f(x_n)$ (residual)

   column 4: $|\alpha - x_n|$ (error)

2. Let α be a fixed point of $g(x)$. Consider the fixed point iteration $x_{n+1} = g(x_n)$ and suppose that $\max|g'(x)| = k < 1$. Prove the following error estimate:

$$|\alpha - x_{n+1}| \leq \frac{k}{1-k}|x_{n+1} - x_n|.$$

3. (a) Kepler's equation in astronomy reads $x = y - \epsilon \sin y$ with $0 < \epsilon < 1$. Show that for each $x \in [0, \pi]$ there is a y satisfying the equation. (Hint: Interpret this as a fixed-point problem).

   (b) Show that the equation

$$x = 5 + 0.25 \sin 2x$$

   has a unique solution α. Show that the iteration $x_{n+1} = 5 + 0.25 \sin 2x_n$ will converge to α. Find a bound for the error.

4. Let α be the solution of $f(x) = 0$, and $\{x_n\}$ be the sequence of approximate solutions, generated by the Newton's method. Show that

$$\alpha - x_{n+1} = -\frac{1}{2}\frac{f''(\xi_n)}{f'(x_n)}(\alpha - x_n)^2,$$

   where ξ_n is between x_n and α.

5. (a) Let p be a positive number. What is the value of the following expression:

$$x = \sqrt{p + \sqrt{p + \dots}}$$

   Note that this can be interpreted as meaning $x = \lim_{n\to\infty} x_n$, where $x_1 = \sqrt{p}$, $x_2 = \sqrt{p + \sqrt{p}}$ and so on (Hint: Observe that $x_{n+1} = \sqrt{p + x_n}$.

   (b) To find a zero of the function f, we can look for a fixed point of the function $F(x) = x - f(x)/f'(x)$. To find a fixed point of F, we can solve $F(x) - x = 0$ by Newton's method. When this is done, what is the formula for generating the sequence x_n?