Math 609, Homework 6

1. Let \( n, k \), the knots \( t_1, \ldots, t_{n+k+1} \), and the coefficients \( c_1, \ldots, c_n \) be specified, and \( f(x) = \sum_{i=1}^n c_i B_i^k(x) \). Write a program that has an input \( n, k \), the knots and the coefficients and as output gives the value of \( f(x) \) for any \( x \). Test the program for several values of your input.

2. (a) Show that for \( k \geq 3 \),

\[
\frac{d^2}{dx^2} \sum_{i=-\infty}^{\infty} c_i B_i^k(x) = k(k-1) \sum_{i=-\infty}^{\infty} \left( \frac{c_i - c_{i-1}}{t_{i+k} - t_i} - \frac{c_{i-1} - c_{i-2}}{t_{i+k-1} - t_{i-1}} \right) \frac{B_i^{k-2}(x)}{t_{i+k} - t_i}
\]

(b) Suppose that the knots are taken to be all the integers, namely \( t_i = i \). Show that \( B_i^k(x) = B_0^k(x - t_i) \).

(c) Show that

\[
\sum_{i=0}^{n} B_i^k(x) = 1
\]

when \( x \in [t_k, t_{k+n}] \).

(d) Show that

\[
\sum_{i=0}^{n} B_i^k(x) > 0
\]

when \( x \in [t_1, t_{k+n+1}] \).

3. (a) Can \( a \) and \( b \) be defined so that the function \( S \), defined as: \( S(x) = (x - 2)^3 + a(x - 1)^2 \), when \( x \leq 2 \), \( S(x) = (x - 2)^3 - (x - 3)^2 \), when \( 2 \leq x \leq 3 \), and \( S(x) = (x - 3)^3 + b(x - 2)^2 \), when \( x \geq 3 \) is a natural cubic spline? Why or why not?

(b) Find a natural cubic spline function \( S \) with knots \(-1, 0, 1\) that interpolates the table:

\[
S(-1) = 5, \quad S(0) = 7, \quad S(1) = 9.
\]

4. Let

\[
f(x) = \frac{1}{1 + 6x^2}
\]

(a) Write a program to perform Lagrange polynomial interpolation at the uniform points and the Chebyshev points on the interval \([-1, 1]\) for \( f \).

(b) Write a program to perform natural cubic spline interpolation at the uniform points for \( f \).

Investigate the convergence by running the program for different values of \( n \). In the write up include plots of \( f, L_n, S_n \) for \( n = 8, 16, 32 \). Answer the following questions.

Does \( L_n \) converge uniformly to \( f \) on \([-1, 1]\)?

Does \( S_n \) converge uniformly to \( f \) on \([-1, 1]\)?