Math 417, Homework 8

1. (a) Can a and b be defined so that the function S, defined as: $S(x) = (x-2)^3 + a(x-1)^2$, when $x \le 2$, $S(x) = (x-2)^3 - (x-3)^2$, when $2 \le x \le 3$, and $S(x) = (x-3)^3 + b(x-2)^2$, when $x \ge 3$ is a natural cubic spline? Why or why not?

   (b) Find a natural cubic spline function S with knots -1, 0 and 1 that interpolates the table:

$$S(-1) = 5, \quad S(0) = 7, \quad S(1) = 9.$$

2. Let

$$f(x) = \frac{1}{1+6x^2}$$

   (a) Write a program to perform Lagrange polynomial interpolation at the uniform points and the Chebyshev points on the interval $[-1,1]$ for f.

   (b) Write a program to perform natural cubic spline interpolation at the uniform points for f.

   Investigate the convergence by running the program for different values of n. In the write up include plots of f, L_n, S_n for $n = 8$, 16, 32. Answer the following questions.

   Does L_n converge uniformly to f on $[-1,1]$?

   Does S_n converge uniformly to f on $[-1,1]$?

3. The local form of the midpoint rule is

$$\int_0^h f(x)\,dx \approx cf(\frac{h}{2}).$$

   (a) Determine the value of the constant c which ensures that the midpoint rule is exact for constant functions. Show that the method is actually exact for linear polynomials.

   (b) Is the midpoint rule more accurate or less accurate than the trapezoid rule? Justify your answer.

4. Consider the integral

$$I = \int_0^1 x^2 e^{-x^2}\,dx.$$

   (a) Suppose that I is approximated using the trapezoid rule. Use the error estimate derived in class to determine a value of h which will ensure that the error is less than 10^{-6}.

   (b) Compute approximate values of I using the trapezoid rule and Richardson extrapolation with $h = 1.0$, 0.5, 0.25, 0.125. Present your results in the form of a table, as in the example from class.

5. Prove that

   (a) If

$$\int_a^b f(x)\omega(x)\,dx = \sum_{i=0}^{n} A_i f(x_i)$$

   for all polynomials of degree $2n+1$, then the polynomial $(x-x_0)\cdots(x-x_n)$ is orthogonal to π_n on $[a,b]$ with respect to ω.

   (b) No Gaussian quadrature formula with n nodes can be exact on π_{2n}.