

Central-upwind schemes for hyperbolic conservation laws

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1 Introduction

In the last decade, Godunov-type central schemes emerge as simple, reliable, efficient, high-resolution methods for solving time-dependent PDE's with wide applications. They do not employ Riemann problem solvers and characteristic decomposition which makes them an attractive alternative to problem oriented upwind methods.

Examples of recently developed Godunov-type central schemes are the so-called semi-discrete central-upwind schemes, where a more careful estimate of the one-sided local speeds of propagation and integration over Riemann fans with variable sizes is used. This decreases the numerical dissipation and results in increased resolution of the computed solution. Another advantage of these schemes, as opposed to the earlier developed staggered central schemes, is that they can be used for steady state computations.

Here, we illustrate the potential of the second order semi-discrete central-upwind schemes for computing solutions to the Euler equations of gas dynamics with non-convex equation of state which is a challenging problem because of the formation of composite waves. We also demonstrate that these schemes can be applied to problems with complex geometries, where the use of triangular or mixed rectangular-triangular grids is favorable.

2 Semi-discrete central-upwind schemes

Godunov-type schemes are used to solve the hyperbolic system of conservation laws,

$$u_t + \nabla_{\mathbf{x}} \cdot f(u) = 0, \quad (1)$$

subject to the initial data

$$u(\mathbf{x}, 0) = u_0(\mathbf{x}).$$

They consist of three steps: projection, reconstruction and evolution (for the sake of simplicity, we consider the one-dimensional case, and let Δt and Δx be small time and spacial scales, respectively, and $(x_j, t^n) := (j\Delta x, n\Delta t)$ be a uniform grid).

- The projection step is the averaging of the solution over the computational cells, i.e. the computing of

$$\bar{u}(x, t) := \frac{1}{\Delta x} \int_{I(x)} u(\xi, t) d\xi, \quad I(x) := \{\xi : |\xi - x| < \Delta x/2\}.$$

- The reconstruction step is the construction of a high order piecewise polynomial interpolant \tilde{u} from the cell averages,

$$\tilde{u}(x, t^n) = p_j^n(x), \quad x_{j-\frac{1}{2}} < x < x_{j+\frac{1}{2}}, \quad \forall j. \quad (2)$$

The interpolant should be conservative, sufficiently accurate and non-oscillatory. The latter requirement is typically achieved with the help of nonlinear limiters. The choice of \tilde{u} is crucial in computing the entropy solutions of hyperbolic conservation laws with non-convex fluxes or of hyperbolic systems of conservation laws with non-convex equations of state.

- The evolution step is the evolution of the interpolant to the next time level according to the integral form of (1), which in the one-dimensional case is

$$\bar{u}(x, t + \Delta t) = \bar{u}(x, t) - \frac{1}{\Delta x} \left[\int_{\tau=t}^{t+\Delta t} f\left(u\left(x + \frac{\Delta x}{2}, \tau\right)\right) d\tau - \int_{\tau=t}^{t+\Delta t} f\left(u\left(x - \frac{\Delta x}{2}, \tau\right)\right) d\tau \right]. \quad (3)$$

The possible discontinuities of the piecewise polynomial \tilde{u} (see (2)) at the interface points $\{x_{j+\frac{1}{2}}\}$ propagate with *right-* and *left-sided* local speeds, $a_{j+\frac{1}{2}}^{\pm}$. In the convex case, they can be estimated by

$$\begin{aligned} a_{j+\frac{1}{2}}^+ &= \max \left\{ \lambda_N \left(\frac{\partial f}{\partial u}(u_{j+\frac{1}{2}}^-) \right), \lambda_N \left(\frac{\partial f}{\partial u}(u_{j+\frac{1}{2}}^+) \right), 0 \right\}, \\ a_{j+\frac{1}{2}}^- &= \min \left\{ \lambda_1 \left(\frac{\partial f}{\partial u}(u_{j+\frac{1}{2}}^-) \right), \lambda_1 \left(\frac{\partial f}{\partial u}(u_{j+\frac{1}{2}}^+) \right), 0 \right\}, \end{aligned}$$

where $\lambda_1 < \dots < \lambda_N$ are the N eigenvalues of the Jacobian $\frac{\partial f}{\partial u}$, and $u_{j+\frac{1}{2}}^+ := p_{j+1}^n(x_{j+\frac{1}{2}})$ and $u_{j+\frac{1}{2}}^- := p_j^n(x_{j+\frac{1}{2}})$ are the corresponding right and left values of the reconstruction. In the non-convex case, a more careful estimation is required (see [5, 3] for details).

The cell averages on the next level $t = t^{n+1}$ are obtained from (3) by integrating over non-uniform rectangular domains, which after an intermediate reconstruction are projected back onto the original grid. If we take the semi-discrete limit ($\Delta t \rightarrow 0$), we obtain the semi-discrete central-upwind scheme

$$\frac{d}{dt} \bar{u}_j(t) = - \frac{H_{j+\frac{1}{2}}(t) - H_{j-\frac{1}{2}}(t)}{\Delta x}, \quad (4)$$

where the numerical fluxes $H_{j+\frac{1}{2}}$ are given by

$$H_{j+\frac{1}{2}}(t) := \frac{a_{j+\frac{1}{2}}^+ f(u_{j+\frac{1}{2}}^-) - a_{j+\frac{1}{2}}^- f(u_{j+\frac{1}{2}}^+)}{a_{j+\frac{1}{2}}^+ - a_{j+\frac{1}{2}}^-} + \frac{a_{j+\frac{1}{2}}^+ a_{j+\frac{1}{2}}^-}{a_{j+\frac{1}{2}}^+ - a_{j+\frac{1}{2}}^-} \left[u_{j+\frac{1}{2}}^+ - u_{j+\frac{1}{2}}^- \right].$$

Formula (4) is a system of time-dependent ODEs, which should be solved by a stable, sufficiently accurate ODE solver (see [1]).

3 Numerical examples

In this section, we solve numerically the one-dimensional Euler equations of gas dynamics

$$\frac{\partial}{\partial t} \begin{bmatrix} \rho \\ m \\ E \end{bmatrix} + \frac{\partial}{\partial x} \begin{bmatrix} m \\ \rho u^2 + p \\ u(E + p) \end{bmatrix} = 0,$$

with the non-convex equation of state

$$p = p(\rho, e) = ((\gamma - 1)\rho + f(\rho))e, \quad e = \frac{E}{\rho} - \frac{1}{2}u^2, \quad \gamma = 1.4,$$

where

$$f(\rho) = \begin{cases} 10e^{-\frac{\rho^2}{1-2\rho+0.75\rho^2}}, & \frac{2}{3} < \rho < 2, \\ 0, & \text{otherwise,} \end{cases}$$

and ρ , u , $m = \rho u$, p and E are the density, velocity, momentum, pressure, and total energy respectively. We solve the Riemann problem with initial data

$$(p, \rho, u)_L = (2.7, 2, -0.9), \quad (p, \rho, u)_R = (2.7, 2, 0.9).$$

The pressure and velocity at time $T = 0.34$ are presented in Figure 1 and Figure 2.

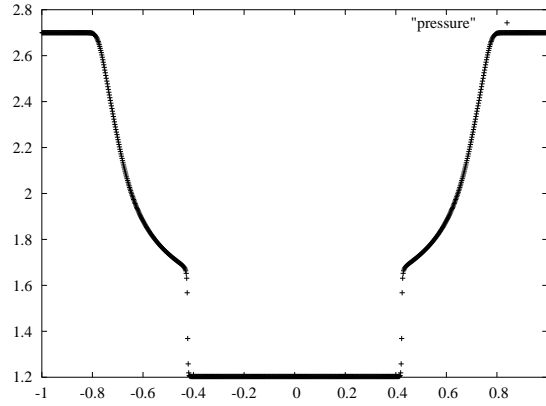


Figure 1: Pressure, $N = 1000$ points, CFL=0.1.

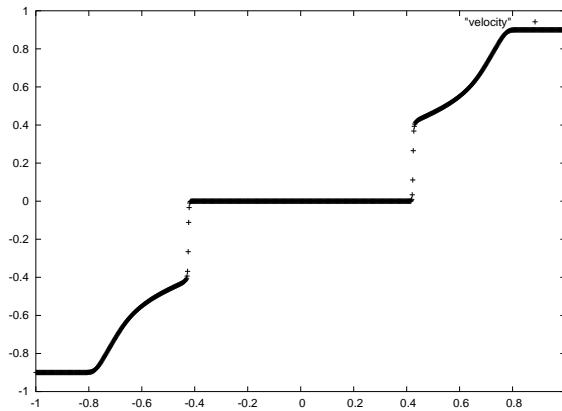


Figure 2: Velocity, $N = 1000$ points, CFL=0.1.

Next, we use two-dimensional versions (see [1, 2]) of the semi-discrete scheme (4) to solve the two-dimensional Euler equations of gas dynamics for ideal gases

$$\frac{\partial}{\partial t} \begin{bmatrix} \rho \\ \rho u \\ \rho v \\ E \end{bmatrix} + \frac{\partial}{\partial x} \begin{bmatrix} \rho u \\ \rho u^2 + p \\ \rho uv \\ u(E + p) \end{bmatrix} + \frac{\partial}{\partial y} \begin{bmatrix} \rho v \\ \rho uv \\ \rho v^2 + p \\ v(E + p) \end{bmatrix} = 0, \quad p = (\gamma - 1) \left[E - \frac{\rho}{2}(u^2 + v^2) \right],$$

on trapezoidal domain. Here ρ , u , v , p , and E are the density, the x - and y -velocities, the pressure, and the energy, respectively.

We consider the problem, describing the shock reflection by a wedge of angle $\theta = \arctan\left(\frac{1}{3}\right)$. The initial conditions correspond to a right-moving Mach 2 shock, initially positioned to the left

of the wedge. Contour plots of the density after the reflection are presented in Figure 3. We have used rectangular computational cells along the lower side of the trapezoid and triangular cells along its right side. The boundary conditions along the lower and right parts of the boundary are treated as a solid wall via the ghost cell technique.

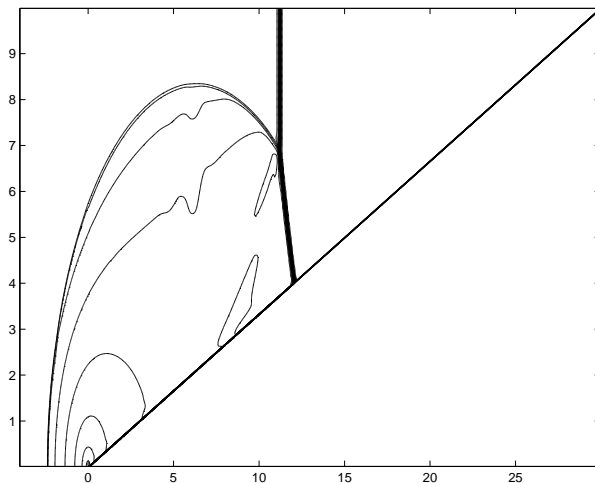


Figure 3: Density contours.

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