Definition 1

Negation If p is a statement, the statement $\sim p$ is the negation of p.

Example 1 Form the negation of the following statements.

- **a** The moon is rising.
- **b** $\angle ABC$ is a remote interior angle.
- **c** Point C is between points A and B.
- **d** $m\angle 3 = 25°$

Solution

- **a** The moon is not rising.
- **b** $\angle ABC$ is not a remote interior angle.
- **c** Point C is not between points A and B.
- **d** $m\angle 3 \neq 25°$

Definition 2

Conjunction If p and q are statements the compound statement 'p and q' is the conjunction.

Example 2 Indicate whether the following statements and their conjunctions are true or false.

- **a** Dogs have five feet. Cats have six feet.
- **b** Every line contains at least one point. Every plane contains all the points of space.
- **c** Every line contains at least two points. Every plane contains at least three points.

Solutions

- **a** Both of the statements are false, so the conjunction is false.
- **b** The statement about the plane is false so the conjunction is false.
- **c** Both statements are true so the conjunction is true.

Definition 3

Disjunction If p and q are statements then 'p or q' is the disjunction of p and q.

Use the previous pairs of statements and determine if the disjunctions are true or false.
Solutions

- **a** Both of the statements are false so the disjunction is false.
- **b** One of the statements is true so the disjunction is true.
- **c** Both statements are true so the disjunction is true.

Theorems are often stated in if-then form. I.E. If the sky is blue then it is not raining. Or in general 'if p then q'. Where p is called the hypothesis, and q is called the conclusion. Here are some examples of if-then statements. Determine what the hypothesis and conclusion are.

Example 3

- **a** If it is cloudy, then it will rain.
- **b** If I eat too much, I will get sick.
- **c** I will be very upset if I fail geometry.
- **d** All crazy people like porcupines.

Definition 4

Converse If you interchange the hypothesis and the conclusion of a statement then you have formed a converse. Example of a converse of the statement in the previous paragraph If it is not raining then the sky is blue.

Definition 5

Inverse If you negate the hypothesis and the conclusion you have formed the inverse. If the sky is not blue then it is raining, is an example of an inverse.

Definition 6

Contrapositive The converse of the inverse is called the contrapositive (or the inverse of the converse). If it is raining then the sky is not blue.

Example 4 Find the converse, inverse and contrapositive of 'It is raining if the sky is cloudy'.

Solution First put the sentence in standard if-then form, 'If the sky is cloudy then it is raining'.

converse If it is raining then the sky is cloudy.

inverse If the sky is not cloudy then it is not raining.

contrapositive If it is not raining then it is not cloudy.

Homework 1

1. Write the hypothesis and conclusion to each of the following. Also write the converse, inverse and contrapositive.
   (a) If I earn enough money then I will buy a house.
   (b) If you get fat then you ate too much.
   (c) You will be able to study better if your SO (significant other) is not with you.
   (d) A theorem always has a hypothesis and a conclusion.
2. Decide what is given and what is to be proved for each of the following statements.
   (a) The supplements of congruent angles are congruent.
   (b) If two lines intersect, the bisectors of a pair of adjacent angles formed are perpendicular.
   (c) The bisectors of the three angles of a triangle intersect in the same point.

Definition 7

Logical System consists of undefined terms, definitions, assumptions and theorems.

The undefined terms for geometry are **set**, **point**, **line**, **plane**.

Definition 8

One-to-one correspondence is if two sets have the same number of elements. If two sets have a one-to-one correspondence (written 1-1) then they are said to be equivalent.

Definition 9

$\aleph$ is the symbol for the natural numbers. The ones that start with $\{1, 2, 3, \cdots\}$ and continue on.

Definition 10

Finite A set is finite if it is equivalent to a set of the form $\{1, 2, 3, 4 \cdots, m\}$ for some $m \in \aleph$. Otherwise it is infinite.

The basic structure of a geometry contains the set S, the set of points, the set L, the set of lines, and the set P, the set of planes. With just this structure there is no guarantee that we will get the familiar geometry of Euclid. The only thing we will let ourselves know about these sets is in the following postulates. These are called incidence postulates because they define how the various parts of the geometry interact.

***Postulate* 1** All lines and planes are sets of points.

***Postulate* 2** Given any two different points there is exactly one line containing them. If P and Q are two points then the line is denoted $\overleftrightarrow{PQ}$.

***Postulate* 3** Given any three different non-collinear points, there is exactly one plane containing them. If P, Q and R are points then the plane containing them is $\overleftrightarrow{PQR}$.

***Postulate* 4** If two points lie in a plane then the line containing them lies in the plane.

***Postulate* 5** If two planes intersect, then their intersection is a line.

***Postulate* 6** Every line contains at least two points. S contains at least three non-collinear points. Every plane contains at least three non-collinear points. And S contains at least four non-coplanar points.

Note that all of the statements reading two points, two planes, etc. it may occasionally mean that the two things could actually be the same. Though all of the above postulates imply two different items.

***Theorem* 1** Two different lines intersect in at most one point.

***Theorem* 2** If a line intersects a plane not containing it, then the intersection is a single point.

***Theorem* 3** Given a line and a point not on the line, there is exactly one plane containing both of them.

***Theorem* 4** If two different lines intersect, then their union lies in exactly one plane.

Homework 2

1. Consider the space with exactly four points {a,b,c,d}, in the positions of a tetrahedron (triangular pyramid). Verify the incidence postulates.
2. Let p_1, p_2, p_3, p_4, p_5, be five points, no three of which are collinear. How many lines are there? If no four of the five points are coplanar, how many planes are there?
3. Expand the previous two problems to n points. It might help to start with four and then add a point and attempt to find a pattern.
4. Show that under the incedence postulates, the space S cannot be a line.
5. Show that there is at least one plane.
6. Show that there are at least two planes.

An axiomatic system example. Assume that a club of two or more students is organized into committees in such a way that each of the following conditions are satisfied.

a) Every committee is a set of one or more students.

b) For each pair of students, there is exactly one committee on which both serve.

c) No single committee is composed of all students in the club.

d) Given any committee and any student not on that committee, there exists exactly one committee on which that student serves which has no students of the first committee in its membership.

Prove each of the following, justifying each step of your proof to one of the four conditions or one of the statements previously proven.

i Every student serves on at least two committees.

ii Every committee has at least two members.

iii There are at least four students in the club.

iv There are at least six committees in the club.

Add an additional condition.

e) No committee has more than two members.

Prove:

v There are exactly four members in the club.

vi There are exactly six committees in the club.

***Postulate* 7** d is a function from $S \times S \to \Re$

***Postulate* 8** For every P, Q, $d(P,Q) \geq 0$. We will write $d(P,Q)$ as just PQ for ease of use. Don't confuse this with $\overleftrightarrow{PQ}$ which is a line.

***Postulate* 9** $d(P,Q) = 0$ if and only if (iff) $P = Q$.

***Postulate* 10** $d(P,Q) = d(Q,P)$ for every P and Q in S.

***Postulate* 11**

Ruler Postulate [3.3.d-4] Every line has a coordinate system.

Definition 11

Betweenness Let A, B, C be three collinear points. If $AB + BC = AC$ then B is between A and C. We write $A - B - C$.

***Theorem* 5** If $A - B - C$ then $C - B - A$

Lemma 1 Given a line L with a coordinate system and three points A,B, C with y between x and z then $A - B - C$.

***Theorem* 6** Of any three points on a line exactly one is between the other two.

If we write $A - B - C - D$ (with w, x, y, z being coordinates respectively) then either $w < x < y < z$ or $w > x > y > z$. And we extend this notation as necessary for more points.

***Theorem* 7** Any four points of a line can be named such that $A - B - C - D$.

***Theorem* 8** If A and B are any two points, then

1. there is a point C such that $A - B - C$, and
2. there is a point D such that $A - D - B$.

Homework 3

1. Show that if $A - B - C$ and $B - C - D$ then $A - B - D$ and $A - C - D$.
2. Show that if $A - B - C$ and $A - D - C$ then $A - B - D - C$ or $A - D - B - C$ or $B = D$.
3. Given four beads of different colours. In how many different ways is it possible to arrange them in a trough, in order from left to right. How many ways is it possible to arrange them on a rigid symmetrical rod? How many different ways to arange them on a circular loop?

4. Which postulate is illustrated by each of the following, if A,B, C are different points, k and m are two lines, and P is a plane?
   (a) If $A \in P, C \in P, A \in k$ and $C \in k$, then $k \in P$
   (b) If $B \in k, C \in k$ and $k \neq m$ then B and C cannot both be on m.
   (c) If $A \in k, B \in k, A \in m$, and $B \in m$ then $k = m$.
   (d) There is a point C such that $C \notin P$.
   (e) If $A \in k, B \in k$, and $C \notin k$ then there is a plane P such that $A \in P$, $B \in P$ and $C \in P$.

Definition 12

Segment If A and B are two points then the segment between them is the set of points $A - X - B$, for all X in S, along with A and B. It is written $\overline{AB}$

Definition 13

Ray Is the set of points C on $\overleftrightarrow{AB}$ with A not between B and C. It is written $\overrightarrow{AB}$.

Definition 14

Ray (alternate) Is the union of $\overline{AB}$ and the set of all points C such that $A - B - C$.

Definition 15

Angle Is the union of two rays with the same endpoint but don't lie on the same line. If the angle is the union of $\overrightarrow{BC}$ and $\overrightarrow{CD}$ then the rays are called the sides of the angle and C is the vertex. And the angle is denoted $\angle BCD$ or $\angle DCB$, the vertex is always in the middle. There will be times when we will refer to an angle as $\angle C$ if there is no confusion.

Definition 16

Triangle If A, B, C are three non-collinear points, then the set $\overline{AB} \cup \overline{BC} \cup \overline{CA}$ is a triangle.

The three segments $\overline{AB}, \overline{BC}, \overline{CA}$ are sides. And the vertices are A, B, C. The angles of $\triangle ABC$ are $\angle ABC$, $\angle BAC$ and $\angle ACB$.

***Theorem* 9** [3.5.T-1] If A and B are any two points then $\overline{AB} = \overline{BA}$.

***Theorem* 10** [3.5.T-2] If C is a point of $\overrightarrow{AB}$, other than A, then $\overrightarrow{AB} = \overrightarrow{AC}$

***Theorem* 11** [3.5.T-3] If $B_1 \in \overrightarrow{AB}$ and $C_1 \in \overrightarrow{AC}$, and neither are equal to A, then $\angle BAC = \angle B_1AC_1$

***Theorem* 12** [3.5.T-4] If $\overline{AB} = \overline{CD}$, then the points A,B are the same as the points C, D in some order.

***Theorem* 13** [3.5.T-5] If $\triangle ABC = \triangle DEF$, then the points A, B, C are the same as D, E, F in some order.

Definition 17 [Sidedness] Given points R,S and line k. R and S are on the same side of k if $\overline{RS} \cap k = \emptyset$. R and S are on opposite sides of k if $R \notin k$ and $S \notin k$ and $\overline{RS} \cap k \neq \emptyset$.

Definition 18

Interior Given $\angle ABC$ and a point P then P is in the interior of $\angle ABC$ if P and C are on the same side of $\overleftrightarrow{AB}$ and P and A are on the same side of $\overleftrightarrow{CB}$. It is written $int\angle ABC$.

Definition 19

Exterior Given $\angle ABC$ and a point P then P is in the exterior if $P \notin int\angle ABC$ or $P \notin \angle ABC$.

Definition 20

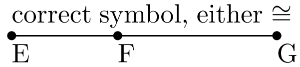
Interior of a triangle Given $\triangle DEF$ then $int\triangle DEF = int\angle DEF \cap int\angle EDF$ (or in general it is the intersection of the interior of any two angles of the triangle.

Definition 21

Congruence of line segments If $AB = DC$ then $\overline{AB} \cong \overline{CD}$.

Homework 4

1. Given $\overleftrightarrow{EC}$ and $\overleftrightarrow{FB}$ with $E - A - C$ and $F - A - B$, list all of the line segments, rays and lines.
2. Given that $AB = EF$ insert the correct symbol, either $\cong$ or $=$ or both.

   (a) $AB?CD$
   (b) $\overline{AB}?\overline{CD}$
   (c) $CD?EF$
   (d) $\overline{CD}?\overline{EF}$
   (e) $CD?EG$
   (f) $\overline{AB}?\overline{EG}$
3. Let A and B be two points, and let D, E, F be three non-collinear points. If $\overleftrightarrow{AB}$ contains only one of the points D, E or F then each of the lines $\overleftrightarrow{DE}$, $\overleftrightarrow{DF}$, $\overleftrightarrow{EF}$ intersects $\overleftrightarrow{AB}$ in at most one point.
4. Show that for any $\triangle ABC$, we have $\overleftrightarrow{AB} \cap \triangle ABC = \overline{AB}$.
5. Show that A is not between any two points of $\triangle ABC$

***Theorem* 14**

Segment construction [3.6.C-2] Given a segment $\overline{AB}$ and a ray $\overrightarrow{CD}$, there is exactly one point E on $\overrightarrow{CD}$ such that $\overline{AB} \cong \overline{CE}$.

***Theorem* 15**

Segment Addition [3.6.C-3] If $A - B - C$ and $D - E - F$ and $\overline{AB} \cong \overline{DE}$ and $\overline{BC} \cong \overline{EF}$ then $\overline{AC} \cong \overline{DF}$.

Definition 22

Convex A set G is convex if for any two points $P, Q \in G$ then $\overline{PQ} \subset G$.

A line separates a plane into two pieces called half-planes. Each half-plane is convex. Each half-plane is bounded on one side by the line that separates them. The line is not part of either half-plane.

***Theorem* 16**

The postulate of Pasch [4.1.T-1] Given $\triangle ABC$, and a line l in the same plane. If l contains a point E between A and C then l intersects either $\overline{AB}$ or $\overline{BC}$.

It turns out that this theorem can be used in place of the sidedness definition. But then the sidedness definition becomes a theorem. So two statements play the same role in geometry.
The following homework follows very nicely from the betweenness theorems. Most of them are very short.
Notation. H_1 and H_2 are half-planes.

Homework 5

1. The half-planes are not both empty. (or one of the half planes has something in it)
2. Neither of the half-planes is empty. (if there is something in one then there is something in the other)
3. H_1 contains at least two points.
4. H_1 contains at least three non-collinear points.
5. E the plane that contains H_1 is unique.
6. If A and B are convex then so is $A \cap B$.
7. If G is a collection of convex sets then the intersection of all of the sets is convex.
8. Every ray is convex.
9. $H_1 \cap l$ is convex.
10. The interior of a triangle is convex. (this one might be impossible with what we know)

These are in the order they ought to be proven. So if you get stuck on one go to the next and assume the one you got stuck on is proved.

***Postulate* 12**

Space separation postulate [4.5.ss-1] Given a plane in space. The set of all points that do not lie in the plane is the union of two sets S_1, S_2 (the book uses H_1 and H_2 we will reserve those for half-planes) such that each of the sets is convex, and if $P \in S_1$ and $Q \in S_2$ then $\overline{PQ}$ intersects the plane.

The two sets S_1 and S_2 are called half-spaces of the plane E that separates them. E is also called the face to each half-space.

***Theorem* 17** The sets S_1 and S_2 are not both empty.

***Theorem* 18** Neither of the sets S_1 and S_2 is empty.

***Theorem* 19** Each of the sets S_1 and S_2 contains at least four non-coplanar points.

***Theorem* 20** E is uniquely determined by S_1 (every half-space has only one face).

***Theorem* 21** Let H be a half plane with edge l, and let E be a plane that contains l but not H. Then all points of H are on the same side of E.

Definition 23

Dihedral angle If two half planes H and G have the same edge l, but do not lie in the same plane, then the set $H \cup G \cup l$ is called a dihedral angle.

Homework 6

1. The interior of a dihedral angle is always a convex set.
2. If P and Q are in different sides of a dihedral angle, then every point between P and Q is in the interior of the dihedral angle.

Angles and their Measure

***Postulate* 13** For every angle $\angle A$, $m\angle A$ is between 0 and 180.

***Postulate* 14**

The angle construction postulate Let $\overrightarrow{AB}$ be a ray on the edge of the half plane H. For every number r between 0 and 180, there is exactly one ray $\overrightarrow{AP}$ in H, with $m\angle PAB = r$.

Definition 24

Congruence of angles If two angles have the same measure then they are congruent.

Definition 25

Complementary Two angles are complementary if the sum of the measures is 90.

Definition 26

Supplementary Two angles are supplementary if the sum of the measures is 180.

Definition 27

Vertical Angles Two angles are called vertical angles if their sides form pairs of opposite sides.

***Theorem* 22**

Vertical Angle Theorem If two angles form a vertical pair then they are congruent.

***Theorem* 23** If two intersecting lines form one right angle then they form 4 right angles.

Two triangles are congruent $\triangle ABC \cong \triangle DEF$ if $\overline{AB} \cong \overline{DE}, \overline{BC} \cong \overline{EF}, \overline{CA} \cong \overline{FD}$, $\angle A \cong \angle D$, $\angle B \cong \angle E$, $\angle C \cong \angle F$.
Which means that to show two triangles are congruent you need to show 6 separate congruence. In the Euclidean plane, if you take two sticks and an angle and put the sticks end to end and separate them by the angle the length between the other ends of the sticks is the same every time. Which leads to the next postulate.

***Postulate* 15**

SAS Given a correspondence between two triangles (not necessarily different triangles). If two sides and the included angle of the first triangle are congruent to the corresponding parts of the second then the triangles are congruent. SAS stands for Side-Angle-Side.

***Theorem* 24**

Isosceles Triangle Theorem [6.2.T-1] If two sides of a triangle are congruent then the angles opposite them are congruent.

Corollary 1 Every equilateral triangle is equiangular.

***Theorem* 25**

ASA [6.2.T-2] If two angles and the included side of the first triangle are congruent to corresponding parts of the second, then the triangles are congruent.

Corollary 2 If two angles of a triangle are congruent then the sides opposite them are congruent.

Corollary 3 Every equiangular triangle is equilateral.

***Theorem* 26**

SSS If all three pairs corresponding sides are congruent then the triangles are congruent.

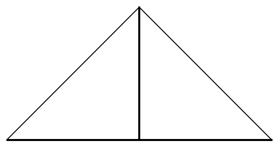
***Theorem* 27** Every angle has exactly one bisector.

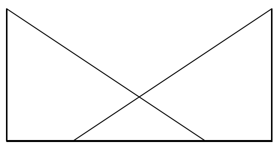
***Theorem* 28**

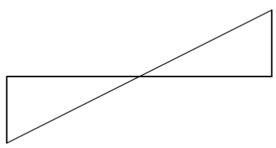
Perpendiculars exist Given a line and a point not on the line, then there is a line which passes through the given point and is perpendicular to the given line.

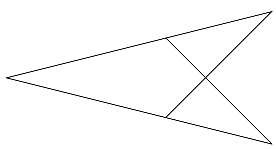
Homework 7

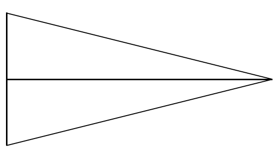
Identify the congruent triangles and show they are congruent. Note a picture can have more than one pair of triangles.

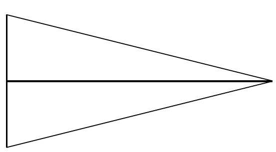
1. 

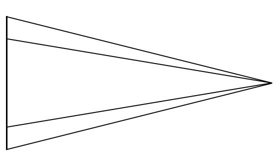
2. 

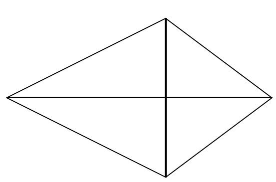
3. 

4. 

5. 

6. 

7. 

8. 

***Theorem* 29** If A and B are equidistant from P and Q then every point between A and B has the same property.

***Theorem* 30** If a line L contains the midpoint of $\overline{PQ}$ and contains another point which is equidistant from P and Q, then $L \perp \overleftrightarrow{PQ}$.

***Theorem* 31** If a line is perpendicular to each of two intersecting lines at their point of intersection, then it is perpendicular to the plane that contains them.

***Theorem* 32** If L is the perpendicular bisector of the segment $\overline{AB}$ then all points of L are equidistant from A and B.

***Theorem* 33** Let A, B and P be points of a plane E. IF P is equidistant from A and B, then P lies on the perpendicular bisector of $\overline{AB}$

***Theorem* 34** The perpendicular bisector of a segment in a plane is the set of all points of the plane that are equidistant from the end points of the segment.

***Theorem* 35** Given a line l and a point P of l. There is only one plane which is perpendicular to l at P.

***Theorem* 36** Every point of the perpendicular bisecting plane of a segment is equidistant from the end points of the segment.

***Theorem* 37** Every point equidistant from the end points of a segment lies in the perpendicular bisecting plane of the segment.

***Theorem* 38** The perpendicular bisecting plane of a segment is the set of all points that are equidistant from the end points of the segment.

***Theorem* 39** Any two lines perpendicular to the same plane are coplanar.

***Theorem* 40** Through a given point in a given plane there is at least one line perpendicular to the given plane.

***Theorem* 41** Through a given point in a given plane there is at most one line perpendicular to the given plane.

***Theorem* 42** Through a given point not in a given plane there is at least one line perpendicular to the given plane.

***Theorem* 43** Through a given point not in a given plane there is at most one line perpendicular to the given plane.

***Theorem* 44** Given a point and a plane, there is exactly one line which passes through the given point and is perpendicular to the given plane.

***Theorem* 45** If a plane E and line l are perpendicular at a point P, then E contains every line that passes through P and is perpendicular to l.

Homework 8

1. Prove the converse of theorem 32.
2. Prove theorem 35. Don't forget to do both parts, that there is one and that there is only one.
3. Prove theorems 40, 41, 42, 43, 44, 45.
4. Show that if a line l contains two points equidistant from P and Q, then every point of l is equidistant from P and Q.
5. Show that if a plane E contains three non-collinear points which are equidistant from P and Q, then all points of E are equidistant from P and Q.

Definition 28

Parallel Two lines are parallel in a plane if they don't intersect.

***Postulate* 16**

Parallel Postulate Through a point not on a given line there is exactly one parallel to the line.

***Postulate* 17**

Lobachevsky and Bolya Postulate Through a point not on a given line there are infinitely many parallels to the line. Poincare came up with a model for this one using circles that are perpendicular to each other.

***Postulate* 18**

Riemann Postulate Through a point not on a given line there are no parallels to the line. This is modeled by great circles on a sphere.

We will be doing something with these later if there is time.

Definition 29

Exterior Angle Given $\triangle ABC$ and $A-B-D$ then $\angle CBD$ is an exterior angle. $\angle A$ and $\angle C$ are the remote interior angles to $\angle CBD$.

Definition 30 $\angle ABC$ is smaller than $\angle DEF$ if there exists a point P in the interior of $\angle DEF$ with $\angle PEF \cong \angle ABC$.

***Theorem* 46** An exterior angle of a triangle is greater in measure than either remote interior angle.

***Theorem* 47** If two angles of a triangle are not congruent, the sides opposite these angles are not congruent.

***Theorem* 48** If two sides of a triangle are not congruent the angles opposite these sides are not congruent, and the angle with the larger measure is opposite the longer side.

***Theorem* 49** If two angles of a triangle are not congruent, the sides opposite these angles are not congruent, and the longer side is opposite the angle with larger measure.

***Theorem* 50** If two lines in a plane are perpendicular to a third line in the same plane then the two lines are parallel.

Homework 9

1. Given triangle $\triangle ABD$ and $A-C-D$.
   (a) Given $m\angle D < m\angle A < m\angle ACB$ Prove $BC < AB < BD$.
   (b) Given $AC = CD, m\angle ABC \neq m\angle CBD$. Prove $m\angle ACB \neq m\angle DCB$.
2. Given $\triangle ABE$ and $A-C-D-E$.
   (a) Given $\triangle ABC \ncong \triangle EBD$ and $AC = DE$. Prove $m\angle BCD \neq m\angle BDC$
   (b) Given $AB = BE, AC \neq DE$. Prove $m\angle ABC \neq m\angle EBD$
   (c) Given $m\angle BCD \neq m\angle BDC$ and $AC = DE$. Prove $AB \neq BE$.
   (d) Given $\angle ACB \cong \angle BDE$ and $m\angle ABC \neq m\angle DBE$. Prove $AC \neq DE$.

Definition 31

Transversal A transversal is a line that intersects two other lines at different points. The angles that contain both intersection points are called interior angles. The other four are called exterior angles. The two nonadjacent interior angles on opposite sides of the transversal are alternate interior angles. The two nonadjacent exterior angles are called alternate exterior angles.

Definition 32 Two nonadjacent angles on the same side of the transversal are corresponding angles if one is interior and one is exterior.

***Theorem* 51** If two lines are cut by a transversal so that alternate interior angles are congruent, then the lines are parallel.

***Theorem* 52** If two parallel lines are cut by a transversal, then the alternate interior angles are congruent.

***Theorem* 53** Corresponding angles formed by two lines and a transversal are congruent if and only if the lines are parallel.

***Theorem* 54** Alternate exterior angles formed by two lines and a transversal are congruent if and only if the lines are parallel.

***Theorem* 55** Interior angles formed by two lines and a transversal, such that the angles are on the same side of the transversal, are supplementary if and only if the lines are parallel.

***Theorem* 56** The sum of the measures of the angles of a triangle is 180.

***Theorem* 57** An exterior angle of a triangle is equal in measure to the sum of the remote interior angles.

Homework 10

1. Given $a \| b$ $D - A - G$ on a and $E - B - C - F$ on b.
   (a) Prove $m\angle DAB + m\angle GAC = m\angle ABC + m\angle ACB$
   (b) Prove $m\angle DAB + m\angle ABE = 180$
   (c) Prove $m\angle ACF > m\angle DAB$
2. Given $\overleftrightarrow{GH} \| \overleftrightarrow{IJ}$ $G - B - E - H$ and $I - A - D - C - F - J$.
   (a) Given $\angle GBA \cong \angle EDC$, prove $\overleftrightarrow{AB} \| \overleftrightarrow{DE}$.
   (b) Given $\angle HEF \cong \angle ACB$, prove $\overleftrightarrow{BC} \| \overleftrightarrow{EF}$.
   (c) Given $\triangle ABC \cong \triangle DEF$, prove $\overleftrightarrow{AB} \| \overleftrightarrow{DE}$ and $\overleftrightarrow{BC} \| \overleftrightarrow{EF}$.
3. Prove $\triangle ABC \cong \triangle DEF$ if $\angle A \cong \angle D$, $\angle C \cong \angle F$ and $\overline{BC} \cong \overline{EF}$.

Definition 33

polygon A polygon is the union of n segments in a plane, intersecting at and only at their endpoints, such that exactly two segments contain each endpoint and no two consecutive segments are on the same line.

We are going to assume that any polygon is also convex unless stated otherwise.

Definition 34

Regular A polygon is regular if all sides and all interior angles are congruent.

Definition 35 A diagonal of a polygon is the line segment joining two non-consecutive vertices.

***Theorem* 58** The sum of the measures of the interior angles of an n-gon is $(n-2)180$.

***Theorem* 59** The sum of the exterior angles of a polygon one at each vertex is 360.

Definition 36

parallelogram is a quadrilateral in which both pairs of opposite sides are parallel.

A rhombus is a parallelogram with a pair of congruent adjacent sides. A rectangle is a parallelogram with a right angle. A square is a rhombus that is a rectangle.

***Theorem* 60** A diagonal divides a parallelogram into two congruent triangles.

***Theorem* 61** If the opposite sides or the opposite angles of a quadrilateral are congruent, the quadrilateral is a parallelogram.

***Theorem* 62** Two parallel lines are everywhere equidistant.

***Theorem* 63** The diagonals of a quadrilateral bisect each other if and only if the quadrilateral is a parallelogram.

***Theorem* 64** Two consecutive angles of a parallelogram are supplementary

Definition 37

Trapezoid Is a quadrilateral with exactly one pair of parallel sides.

Homework 11

1. Two distinct lines parallel to a third line are parallel to each other.
2. If three of more parallel lines intercept congruent segments on one transversal, they intercept congruent segments on every transversal.
3. If a segment joins the midpoints of two sides of a triangle, then it is parallel to the third side and has half the length of the third side.

Definition 38 *Triangular region* is the union of a triangle and its interior.

Definition 39 A *polygonal region* is the union of a finite number of triangular regions such that if two triangular regions intersect, their intersection is an edge or vertex of both.

Definition 40 The *altitude* of a triangle is a perpendicular segment from a vertex to the line containing the opposite side.

***Postulate* 19** α is a function from the set of all polygonal regions to the set of all real numbers.

***Postulate* 20** For every polygonal region R, $\alpha(\mathrm{R})>0$.

***Postulate* 21** If two triangular regions are congruent, then they have the same area.

***Postulate* 22** If two polygonal regions intersect only in edges and vertices, or not at all, then the area of their union is the sum of their areas.

***Postulate* 23** If a square region has edges of length a then its area is a^2.

***Theorem* 65** [*Rectangle Formula*] The area of a rectangular region is the product of its base and its altitude.

***Theorem* 66** The area of a right triangle is half the product of the lengths of its legs.

***Theorem* 67** The area of a triangle is half the product of any base and the corresponding altitude.

***Theorem* 68** The area of a parallelogram is the product of any base and the corresponding altitude.

***Theorem* 69** The area of a trapezoid is half the product of the altitude and the sum of the bases.

***Theorem* 70** If two triangles have the same altitude, then the ratio of their areas is equal to the ratio of their bases.

***Theorem* 71** If two triangles have the same bases, then the ratio of their areas is the ratio of their corresponding altitudes.

Corollary 4 If two triangles have the same base and the same altitude then they have the same area.