Mathematical Probability, Math 411 - Homework 5

From the textbook solve the problem 41at the end of the Chapter 2, and problems 1,2,5,6,7,8 at the end of Chapter 3. Also solve the following problems.

Problem 1. (a) A burglar just stole your key chain which has n keys, exactly two of which open your apartment door. He will try to open your door using the keys one by one in a random order. Whenever a key does not open your door, he will not attempt to use this key again. What is the probability that the burglar opens your door on the kth attempt.

(b) Find the expectation and the variance of the number of tries he has to do to find a key which opens the door. Simplify your answers.
You can use the following formulas:

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2},$$

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

and

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}.$$

Problem 2.

1. Show that for any positive integer n we have

$$\frac{1}{n(n+1)(n+2)} = \frac{1}{2n} - \frac{1}{n+1} + \frac{1}{2(n+2)}$$

2. Use part 1. to find a formula for the sum

$$\sum_{k=1}^{n} \frac{1}{k(k+1)(k+2)}.$$

3. Let X be a random variable with the range $\{1, 2, 3, \dots\}$ and the probability mass function $p_X(n) = \frac{C}{n(n+1)(n+2)}$, where C is some constant. Find the value of C for which this is a well defined probability mass function.

4. Show that $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$ and using this compute $\mathbf{E}(X)$.

5. Show that the variance of X does not exists.

Problem 3. There are n people coming to your birthday party. The probability that the i-th guest brings you a birthday present is p_i independently of all other guests. What is the expected number of presents you get? (Hint: represent as a sum of random variables)