Mathematical Probability, Math 411 - Homework 8

Solve the problems:

Problem 1. Suppose X has PDF function $6(x(1-x)$ for $0 < x < 1$ and 0 otherwise. Find (a) $E[X]$, (b) $E[X^2]$, and (c) $\mathrm{var}(X)$.

Problem 2. Let X be a positive continuous random variable with the probability density function $f(t)$. If $g(t) = tf(t)$ is also a probability density function of some random variable Y what is $\mathbf{E}(X)$? Express $\mathrm{var}(X)$ in terms of $\mathbf{E}(Y)$.

Problem 3. You are operating a train. Ticket for this train costs \$10. The train is late to the destination T minutes, where T is an exponential random variable with parameter 1. If the train is more than 2 minutes and less than 4 minutes late then each customer gets half of the ticket refunded. If the train is more than 4 and less than 10 minutes late each customer gets the full price of the ticket refunded. If the train is more than 10 minutes late each customer gets the full price of the ticket refunded and in addition gets $\$n$. For what values of n would your expected profit be positive? For what values of n would you expect to earn at least \$5 per ticket.

Problem 4. If X is a continuous random variable which attains only positive values (that is $X \geq 0$). Show that

$$\mathbf{E}(X) = \int_0^\infty \mathbf{P}(X \geq t)\ dt.$$

Note that this also holds for discrete random variables, you can try to prove this formula for discrete random variables as well.

Problem 5. If X is an exponential random variable with parameter λ find both PDF and CDF of the random variable $Y = e^X$.

Problem 6. A random variable X is said to have arcsine law if it's CDF is

$$F_X(t) = \frac{2}{\pi} \arcsin(\sqrt{t}).$$

1) What are the possible values that X can have?
2) Find the median of X.
3) Find the PDF of X (you might want to recall some derivatives of typical functions).
4) Find the mean $\mathbf{E}(X)$ (to compute the integral you might want to use the trigonometric substitution and trigonometric double angle identities).

Problem 7. Let A be the set of all pairs (x, y) which satisfy each of the conditions

$$x \geq 0,\ y \geq 0,\ 1 \leq x + y \leq 2.$$

Let the random variables X and Y be jointly continuous with the joint PDF

$$f_{X,Y}(x,y) = \begin{cases} C(x+y), & \text{if } (x,y) \in A, \\ 0, & \text{otherwise.} \end{cases}$$

Find the value of the constant C. Find the marginal PDFs and CDFs. For X, Y find the conditional pdf's.

Problem 8. A food store just got a delivery of 10000 potatoes. It is known that each potato is rotten with probability 0.1. What is the expected number of healthy potatoes? Use the normal approximation to estimate the probability that there are at least 8970 healthy potatoes.

Problem 9. Suppose X and Y have joint PDF $f(x, y) = 6xy^2$ for $0 < x, y < 1$. What is $P(X + Y < 1)$? Find marginal PDF's. Are X and Y independent?

Problem 10. Let the random variables X and Y be jointly continuous with the joint PDF

$$f_{X,Y}(x, y) = \begin{cases} ke^{-(ax+by)}, & \text{if } x > 0, y > 0, \\ 0, & \text{otherwise,} \end{cases}$$

where $a, b > 0$ and k is a constant.

1. Find k.
2. Are X and Y independent?