Concentration of mass on convex bodies

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Abstract

We establish a sharp concentration of mass inequality for isotropic convex bodies: there exists an absolute constant $c > 0$ such that if K is an isotropic convex body in $\mathbb{R}^n$, then

$$\text{Prob}\left(\left\{x \in K : \|x\|_2 \geqslant c\sqrt{n}L_K t\right\}\right) \leqslant \exp\left(-\sqrt{n}t\right)$$

for every $t \geqslant 1$, where L_K denotes the isotropic constant.

1 Introduction

Let K be an isotropic convex body in $\mathbb{R}^n$. This means that K has volume equal to 1, its centre of mass is at the origin and its inertia matrix is a multiple of the identity. Equivalently, there exists a positive constant L_K, the isotropic constant of K, such that

$$\int_K \langle x, \theta\rangle^2 dx = L_K^2 \tag{1.1}$$

for every $\theta \in S^{n-1}$. A major problem in Asymptotic Convex Geometry is whether there exists an absolute constant $c > 0$ such that $L_K \leqslant c$ for every n and every isotropic convex body K in $\mathbb{R}^n$. The best known estimate, due to Bourgain (see [11]), is $L_K \leqslant c\sqrt[4]{n}\log n$, where $c > 0$ is an absolute constant (see [30] for an extension of this estimate to the not-necessarily symmetric case). There is a number of recent developments on this problem; see [13], [14] and [21]. In particular, Klartag in [21] has obtained an isomorphic answer to the question: For every symmetric convex body K in $\mathbb{R}^n$ there exists a second symmetric convex body T in $\mathbb{R}^n$ whose Banach-Mazur distance from K is $O(\log n)$ and its isotropic constant is bounded by an absolute constant: $L_T \leqslant c$.

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The starting point of this paper is the following concentration estimate of Alesker [1]: there exists an absolute constant $c > 0$ such that if K is an isotropic convex body in $\mathbb{R}^n$, then

$$\mathrm{Prob}\left(\{x \in K : \|x\|_2 \geqslant c\sqrt{n}L_K t\}\right) \leqslant 2\exp(-t^2) \tag{1.2}$$

for every $t \geqslant 1$.

Bobkov and Nazarov (see [7] and [8]) have clarified the picture of the volume distribution on isotropic unconditional convex bodies. Recall that a symmetric convex body K is called unconditional if, for every choice of real numbers t_i and every choice of signs $\varepsilon_i \in \{-1, 1\}$, $1 \leqslant i \leqslant n$,

$$\|\varepsilon_1 t_1 e_1 + \cdots + \varepsilon_n t_n e_n\|_K = \|t_1 e_1 + \cdots + t_n e_n\|_K,$$

where $\| \dot{} \|_K$ is the norm that corresponds to K and $\{e_1, \dots e_n\}$ is the standard orthonormal basis of $\mathbb{R}^n$. In particular, they obtained a striking strengthening of (1.2) in the case of 1-unconditional isotropic convex bodies: there exists an absolute constant $c > 0$ such that if K is a 1-unconditional isotropic convex body in $\mathbb{R}^n$, then

$$\mathrm{Prob}\left(\{x \in K : \|x\|_2 \geqslant c\sqrt{n}t\}\right) \leqslant \exp\left(-\sqrt{n}t\right) \tag{1.3}$$

for every $t \geqslant 1$. Note that $L_K \simeq 1$ in the case of 1–unconditional convex bodies (see [27]). Since the circumradius $R(K)$ of an isotropic convex body K in $\mathbb{R}^n$ is always bounded by $(n+1)L_K$ (see [22]), the estimate in (1.3) is stronger than Alesker's estimate for all $t \geqslant 1$. It should be noted that similar very precise estimates on volume concentration were previously given in the case of the ℓ_p^n-balls (see [39], [38], [41] and [40]). Volume concentration for the class of the unit balls of the Schatten trace classes was recently established in [19].

We will prove that an estimate similar to (1.3) holds true in full generality.

Theorem 1.1. *There exists an absolute constant $c > 0$ such that if K is an isotropic convex body in $\mathbb{R}^n$, then*

$$\mathrm{Prob}\left(\{x \in K : \|x\|_2 \geqslant c\sqrt{n}L_K t\}\right) \leqslant \exp\left(-\sqrt{n}t\right) \tag{1.4}$$

for every $t \geqslant 1$.

The proof of Theorem 1.1 is based on the analysis of the growth of the L_q-norms

$$I_q(K) := \left(\int_K \|x\|_2^q dx\right)^{1/q}, \quad (1 \leqslant q \leqslant n) \tag{1.5}$$

of the Euclidean norm $\|\cdot\|_2$ on isotropic convex bodies. It was observed in [32] that Theorem 1.1 follows from the following fact.

Theorem 1.2. *There exists an absolute constant $c > 0$ with the following property: if K is an isotropic convex body in $\mathbb{R}^n$, then*

$$I_q(K) \leqslant c \max\{q, \sqrt{n}\}\, L_K \tag{1.6}$$

for every $2 \leqslant q \leqslant n$.

In fact, it was proved in [32] that Theorem 1.1 is equivalent to the fact that

$$I_q(K) \leqslant c\sqrt{n}\, L_K \tag{1.7}$$

for every $2 \leqslant q \leqslant \sqrt{n}$. An equivalent formulation of this last statement may be given in terms of the function

$$f_K(t) := \int_{S^{n-1}} |K \cap (\theta^{\perp} + t\theta)|\, d\sigma(\theta) \qquad (t \geqslant 0). \tag{1.8}$$

It has been conjectured that f_K is close to the centered Gaussian density of variance L_K^2. This conjecture can be stated precisely in several different ways (see [10], [3]) and has been verified only for some special classes of bodies. It was proved in [32] that (1.7) is equivalent with the following:

Theorem 1.3. *There exist absolute constants $c_1, c_2 > 0$ such that if K is an isotropic convex body in $\mathbb{R}^n$, then*

$$f_K(t) \leqslant \frac{c_1}{L_K} \exp\left(-c_2 \frac{t^2}{L_K^2}\right) \tag{1.9}$$

for every $0 < t \leqslant \sqrt[4]{n} L_K$.

The paper is organized as follows: In Section 2 we show how one can derive Theorem 1.1 from Theorem 1.2 (the argument appears in [32] and [33], but we reproduce it here so that the presentation will be self-contained). Our main tool is the study of the L_q–centroid bodies of K; the q–th centroid body $Z_q(K)$ has support function

$$h_{Z_q(K)}(y) = \left(\int_K |\langle x, y\rangle|^q dx\right)^{1/q}. \tag{1.10}$$

Sections 3, 4 and 5 are devoted to an analysis of this family of bodies, which leads to Theorem 1.2. In fact, our method of proof works for an arbitrary convex body K in $\mathbb{R}^n$, and leads to the following estimate:

Theorem 1.4. *Let K be a convex body in $\mathbb{R}^n$, with volume 1 and center of mass at the origin. Write K in the form $K = T(\tilde{K})$, where $\tilde{K}$ is isotropic and $T \in SL(n)$ is positive definite. Then,*

$$\mathrm{Prob}\left(\{x \in K : \|x\|_2 \geqslant cI_2(K)t\}\right) \leqslant \exp\left(-\frac{\|T\|_{HS}}{\lambda_1(T)} t\right) \tag{1.11}$$

for every $t \geqslant 1$, where $c > 0$ is an absolute constant (we write $\|T\|_{HS}$ for the Hilbert–Schmidt norm and $\lambda_1(T)$ for the largest eigenvalue of T).

In other words, the concentration estimate of Theorem 1.1 is stable: if $\tilde{K}$ is isotropic and if $\|T\|_{HS}/\lambda_1(T)$ is not small, then one has strong concentration for $T(\tilde{K})$.

As a by-product of our method, in Section 6 we obtain a precise estimate for the volume of the L_q-centroid bodies of a convex body. The lower bound in the next Theorem is a consequence of the L_q affine isoperimetric inequality of Lutwak, Yang and Zhang (see [26]).

Theorem 1.5. *Let K be a convex body in $\mathbb{R}^n$, with volume 1 and center of mass at the origin. For every $2 \leqslant q \leqslant n$ we have that*

$$c_1\sqrt{q/n} \leqslant |Z_q(K)|^{1/n} \leqslant c_2\sqrt{q/n}\, L_K, \tag{1.12}$$

where $c_1, c_2 > 0$ are absolute constants.

In Section 7 we apply our concentration estimate to a question of Kannan, Lovász and Simonovits which has its origin in the problem of finding a fast algorithm for the computation of the volume of a given convex body: The isotropic condition (1.1) may be equivalently written in the form

$$I = \frac{1}{L_K^2}\int_K x \otimes x dx, \tag{1.13}$$

where I is the identity operator. Let $\varepsilon \in (0,1)$ and consider N independent random points $x_1, \dots, x_N$ uniformly distributed in K. The question is to find N_0, as small as possible, for which the following holds true: if $N \geqslant N_0$ then with probability greater than $1-\varepsilon$ one has

$$\left\| I - \frac{1}{NL_K^2}\sum_{i=1}^N x_i \otimes x_i \right\|_2 \leqslant \varepsilon. \tag{1.14}$$

Kannan, Lovász and Simonovits (see [23]) proved that one can choose $N_0 = c(\varepsilon)n^2$ for some constant $c(\varepsilon) > 0$ depending only on ε. This was later improved to $N_0 \simeq c(\varepsilon)n(\log n)^3$ by Bourgain [12] and to $N_0 \simeq c(\varepsilon)n(\log n)^2$ by Rudelson [36]. One can actually check (see [17]) that this last estimate can be obtained by Bourgain's argument if we also use Alesker's concentration inequality. See also [20] for an extension of this result. In [18] it was observed that $N_0 \geqslant c(\varepsilon)n\log n$ is enough for the class of unconditional isotropic convex bodies. Theorem 1.1 allows us to prove the same fact in full generality.

Theorem 1.6. *Let $\varepsilon \in (0,1)$. Assume that $n \geqslant n_0$ and let K be an isotropic convex body in $\mathbb{R}^n$. If $N \geqslant c(\varepsilon)n\log n$, where $c > 0$ is an absolute constant, and if $x_1, \dots, x_N$ are independent random points uniformly distributed in K, then with probability greater than $1-\varepsilon$ we have*

$$(1-\varepsilon)L_K^2 \leqslant \frac{1}{N}\sum_{i=1}^N \langle x_i, \theta\rangle^2 \leqslant (1+\varepsilon)L_K^2, \tag{1.15}$$

for every $\theta \in S^{n-1}$.

G. Aubrun has recently proved (see [2]) that in the unconditional case, only $C(\varepsilon)n$ random points are enough in order to obtain $(1+\varepsilon)$–approximation of the identity operator as in Theorem 1.6.

All the previous results remain valid if we replace Lebesgue measure on an isotropic convex body by an arbitrary isotropic log-concave measure. In the last Section of the paper, we briefly discuss this extension.

Notation. We work in $\mathbb{R}^n$, which is equipped with a Euclidean structure $\langle\cdot,\cdot\rangle$. We denote by $\|\cdot\|_2$ the corresponding Euclidean norm, and write B_2^n for the Euclidean unit ball, and S^{n-1} for the unit sphere. Volume is denoted by $|\cdot|$. We write ω_n for the volume of B_2^n and σ for the rotationally invariant probability measure on S^{n-1}. The Grassmann manifold $G_{n,k}$ of k-dimensional subspaces of $\mathbb{R}^n$ is equipped with the Haar probability measure $\mu_{n,k}$.

A convex body is a compact convex subset C of $\mathbb{R}^n$ with non-empty interior. We say that C is symmetric if $x\in C\Rightarrow -x\in C$. We say that C has centre of mass at the origin if $\int_C\langle x,\theta\rangle dx=0$ for every $\theta\in S^{n-1}$. The support function $h_C:\mathbb{R}^n\to\mathbb{R}$ of C is defined by $h_C(x)=\max\{\langle x,y\rangle : y\in C\}$. The gauge function $r_C:\mathbb{R}^n\to\mathbb{R}$ of C is defined by $r_C(x)=\min\{\lambda\geqslant 0 : x\in\lambda C\}$. The mean width of C is defined as $2w(C)$, where

$$w(C)=\int_{S^{n-1}}h_C(\theta)\sigma(d\theta). \tag{1.16}$$

The circumradius of C is the quantity $R(C)=\max\{\|x\|_2 : x\in C\}$, and the polar body C° of C is

$$C^\circ := \{y\in\mathbb{R}^n : \langle x,y\rangle\leqslant 1 \text{ for all } x\in C\}. \tag{1.17}$$

Whenever we write $a\simeq b$, we mean that there exist absolute constants $c_1,c_2>0$ such that $c_1a\leqslant b\leqslant c_2a$. The letters c,c',c_1,c_2 etc. denote absolute positive constants which may change from line to line. We refer to the books [37], [28] and [34] for basic facts from the Brunn-Minkowski theory and the asymptotic theory of finite dimensional normed spaces.

2 Reduction to the behavior of moments

Let K be a convex body of volume 1 in $\mathbb{R}^n$. For every $q\geqslant 1$ we consider the q-th moment of the Euclidean norm

$$I_q(K)=\left(\int_K\|x\|_2^q dx\right)^{1/q} \tag{2.1}$$

and, for every $q\geqslant 1$ and $y\in\mathbb{R}^n$, we set

$$I_q(K,y)=\left(\int_K|\langle x,y\rangle|^q dx\right)^{1/q}. \tag{2.2}$$

Recall that, as a consequence of Borell's lemma (see [28, Appendix III]) one has the following Khintchine–type inequalities.

Lemma 2.1. *Let K be a convex body of volume 1 in $\mathbb{R}^n$. For every $y \in \mathbb{R}^n$ and every $p, q \geqslant 1$ we have that*

$$I_{pq}(K,y) \leqslant c_1 q I_p(K,y) \tag{2.3}$$

where $c_1 > 0$ is an absolute constant. In particular, for every $y \in \mathbb{R}^n$ and every $q \geqslant 2$ we have that

$$I_q(K,y) \leqslant (c_1/2) q I_2(K,y). \tag{2.4}$$

Also, for every $p, q \geqslant 1$ we have that

$$I_{pq}(K) \leqslant c_1 q I_p(K). \tag{2.5}$$

Alesker's concentration estimate (1.2) is equivalent to the following statement.

Theorem 2.2 (Alesker [1]). *Let K be an isotropic convex body in $\mathbb{R}^n$. For every $q \geqslant 2$ we have that*

$$I_q(K) \leqslant c_2 \sqrt{q} I_2(K) \tag{2.6}$$

where $c_2 > 0$ is an absolute constant.

We will prove the following fact.

Theorem 2.3. *There exist universal constants $c_3, c_4 > 0$ with the following property: if K is an isotropic convex body in $\mathbb{R}^n$, then*

$$I_q(K) \leqslant c_4 I_2(K) \tag{2.7}$$

for every $q \leqslant c_3\sqrt{n}$.

Theorem 1.2 is a direct consequence of Theorem 2.3, Lemma 3.9 and Lemma 3.11. Also, in [32] it was proved that Theorem 1.1 is equivalent to the fact that the q-th moments of the Euclidean norm stay bounded (and equivalent to $I_2(K)$) for large values of q. For completeness we show how one can derive Theorem 1.1 from Theorem 2.3.

Proof of Theorem 1.1. Let K be an isotropic convex body in $\mathbb{R}^n$. Fix $q \geqslant 2$. Markov's inequality shows that

$$\mathrm{Prob}(x \in K : \|x\|_2 \geqslant e^3 I_q(K)) \leqslant e^{-3q}. \tag{2.8}$$

From Borell's lemma (see [28, Appendix III]) we get

$$\begin{aligned} \mathrm{Prob}(x \in K : \|x\|_2 \geqslant e^3 I_q(K) s) &\leqslant (1 - e^{-3q}) \left(\frac{e^{-3q}}{1 - e^{-3q}} \right)^{(s+1)/2} \\ &\leqslant e^{-qs} \end{aligned}$$

for every $s \geqslant 1$. Choosing $q = c_3\sqrt{n}$, and using (2.7), we get

$$\mathrm{Prob}(x \in K : \|x\|_2 \geqslant c_4 e^3 I_2(K) s) \leqslant e^{-c_3\sqrt{n} s} \tag{2.9}$$

for every $s \geqslant 1$. Since K is isotropic, we have $I_2(K) = \sqrt{n} L_K$. This proves Theorem 1.1.

3 L_q-centroid bodies

Let K be a convex body of volume 1 in $\mathbb{R}^n$. For $q \geqslant 1$ we define the L_q–centroid body $Z_q(K)$ of K by its support function:

$$h_{Z_q(K)}(y) = I_q(K, y) := \left(\int_K |\langle x, y\rangle|^q dx \right)^{1/q}. \tag{3.1}$$

Since $|K| = 1$, it is easy to check that $Z_1(K) \subseteq Z_p(K) \subseteq Z_q(K) \subseteq Z_\infty(K)$ for every $1 \leqslant p \leqslant q \leqslant \infty$, where $Z_\infty(K) = \mathrm{conv}\{K, -K\}$.

Observe that $Z_q(K)$ is always symmetric, and $Z_q(TK) = T(Z_q(K))$ for every $T \in SL(n)$ and $q \in [1, \infty]$. Also, if K has its center of mass at the origin, then $Z_q(K) \supseteq cZ_\infty(K)$ for all $q \geqslant n$, where $c > 0$ is an absolute constant.

L_q–centroid bodies have appeared in the literature under a different normalization. If K is a convex body in $\mathbb{R}^n$ and $1 \leqslant q < \infty$, the body $\Gamma_q(K)$ was defined in [25] by

$$h_{\Gamma_q(K)}(y) = \left(\frac{1}{c_{n,q}|K|} \int_K |\langle x, y\rangle|^q dx \right)^{1/q},$$

where

$$c_{n,q} = \frac{\omega_{n+q}}{\omega_2 \omega_n \omega_{q-1}}.$$

In other words, $Z_q(K) = c_{n,q}^{1/q} \Gamma_q(K)$ if $|K| = 1$. The normalization in [25] is chosen so that $\Gamma_q(B_2^n) = B_2^n$ for every q. Lutwak, Yang and Zhang (see [26] and [15] for a different proof) have established the following L_q affine isoperimetric inequality.

Theorem 3.1. *Let K be a convex body of volume 1 in $\mathbb{R}^n$. For every $q \geqslant 1$,*

$$|\Gamma_q(K)| \geqslant 1,$$

with equality if and only if K is a centered ellipsoid of volume 1.

Now, for every $p, q \geqslant 1$ we define

$$w_p(Z_q(K)) = \left(\int_{S^{n-1}} h^p_{Z_q(K)}(\theta)\sigma(d\theta) \right)^{1/p}. \tag{3.2}$$

Observe that $w_1(Z_q(K)) = w(Z_q(K))$.

The q-th moments of the Euclidean norm on K are related to the L_q-centroid bodies of K through the following Lemma.

Lemma 3.2. *Let K be a convex body of volume 1 in $\mathbb{R}^n$. For every $q \geqslant 1$ we have that*

$$w_q(Z_q(K)) = a_{n,q} \sqrt{\frac{q}{q+n}} I_q(K) \tag{3.3}$$

where $a_{n,q} \simeq 1$.

Proof. For every $x \in \mathbb{R}^n$ we have (see [31])

$$\left(\int_{S^{n-1}} |\langle x, \theta\rangle|^q \sigma(d\theta)\right)^{1/q} = a_{n,q}\frac{\sqrt{q}}{\sqrt{q+n}}\|x\|_2, \tag{3.4}$$

where $a_{n,q} \simeq 1$. Since

$$w_q(Z_q(K)) = \left(\int_{S^{n-1}}\int_K |\langle x, \theta\rangle|^q dx\sigma(d\theta)\right)^{1/q}, \tag{3.5}$$

the Lemma follows. □

Remark. It is not hard to check that $a_{n,2} = \sqrt{(n+2)/(2n)}$ and

$$I_2(K) = \sqrt{n}w_2(Z_2(K)). \tag{3.6}$$

Definition 3.3. Let C be a symmetric convex body in $\mathbb{R}^n$ and let $\|x\|_C$ be the norm induced on $\mathbb{R}^n$ by C. Set

$$M(C) = \int_{S^{n-1}} \|\theta\|_C d\sigma(\theta) \quad \text{and} \quad b(C) = \max_{x\in S^{n-1}} \|x\|_C.$$

More generally, for every $q \geqslant 1$ set

$$M_q(C) = \left(\int_{S^{n-1}} \|\theta\|_C^q d\sigma(\theta)\right)^{1/q}. \tag{3.7}$$

Define $k_*(C)$ as the largest positive integer $k \leqslant n$ for which

$$\mu_{n,k}\left(F \in G_{n,k} : \tfrac{1}{2}M(C)\|x\|_2 \leqslant \|x\|_C \leqslant 2M(C)\|x\|_2,\ \forall x \in F\right) \geqslant \frac{n}{n+k}. \tag{3.8}$$

The critical dimension k_* is completely determined by the global parameters M and b.

Fact 3.4 (Milman–Schechtman [29]). *There exist $c_1, c_2 > 0$ such that*

$$c_1 n\frac{M(C)^2}{b(C)^2} \leqslant k_*(C) \leqslant c_2 n\frac{M(C)^2}{b(C)^2} \tag{3.9}$$

for every symmetric convex body C in $\mathbb{R}^n$.

We will make essential use of the following result of Litvak, Milman and Schechtman [24]:

Fact 3.5. *There exist $c_1, c_2, c_3 > 0$ such that for every symmetric convex body C in $\mathbb{R}^n$ we have:*

(i) *If $1 \leqslant q \leqslant k_*(C)$ then $M(C) \leqslant M_q(C) \leqslant c_1 M(C)$.*

(ii) *If $k_*(C) \leqslant q \leqslant n$ then $c_2\sqrt{q/n}\, b(C) \leqslant M_q(C) \leqslant c_3\sqrt{q/n}\, b(C)$.*

On observing that $M(C^\circ) = w(C)$ and $b(C^\circ) = R(C)$, we can translate Fact 3.5 as follows:

Lemma 3.6. *There exist $c_1, c_2, c_3 > 0$ such that for every symmetric convex body C in $\mathbb{R}^n$ we have:*

(i) *If $1 \leqslant q \leqslant k_*(C^\circ)$ then $w(C) \leqslant w_q(C) \leqslant c_1 w(C)$.*

(ii) *If $k_*(C^\circ) \leqslant q \leqslant n$ then $c_2\sqrt{q/n}\, R(C) \leqslant w_q(C) \leqslant c_3\sqrt{q/n}\, R(C)$.*

Definition 3.7. Let K be a convex body of volume 1 in $\mathbb{R}^n$. We define

$$q_*(K) = \max\{q \in \mathbb{N} : k_*(Z_q^\circ(K)) \geqslant q\}, \tag{3.10}$$

where $Z_q^\circ(K) := (Z_q(K))^\circ$.

We will need a lower estimate for $q_*(K)$. This depends on the "ψ_α-behavior" of linear functionals on K.

Definition 3.8. Let K be a convex body of volume 1 in $\mathbb{R}^n$ and let $\alpha \in [1, 2]$. We say that K is a ψ_α-body with constant b_α if

$$\left(\int_K |\langle x, \theta\rangle|^q dx\right)^{1/q} \leqslant b_\alpha q^{1/\alpha} \left(\int_K |\langle x, \theta\rangle|^2 dx\right)^{1/2} \tag{3.11}$$

for all $q \geqslant 2$ and all $\theta \in S^{n-1}$. Equivalently, if

$$Z_q(K) \subseteq b_\alpha q^{1/\alpha} Z_2(K) \tag{3.12}$$

for all $q \geqslant 2$. Observe that if K is a ψ_α-body with constant b_α, then $T(K)$ is a ψ_α-body with the same constant, for every $T \in SL(n)$. Also, from (3.12) we see that

$$R(Z_q(K)) \leqslant b_\alpha q^{1/\alpha} R(Z_2(K)) \tag{3.13}$$

for all $q \geqslant 2$.

An immediate consequence of Lemma 2.1 is that there exists an absolute constant $c > 0$ such that every convex body K in $\mathbb{R}^n$ is a ψ_1-body with constant c.

Lemma 3.9. *There exist absolute constants $c_1, c_2 > 0$ such that if K is a convex body of volume 1 in $\mathbb{R}^n$ then, for every $n \geqslant q \geqslant q_*(K)$,*

$$c_1 R(Z_q(K)) \leqslant I_q(K) \leqslant c_2 R(Z_q(K)). \tag{3.14}$$

In particular, if K is an isotropic ψ_α-body with constant b_α then, for every $n \geqslant q \geqslant q_*(K)$,

$$I_q(K) \leqslant c_2 b_\alpha q^{1/\alpha} L_K. \tag{3.15}$$

Proof. Let $n \geqslant q \geqslant q_*(K)$. By the definition of $q_*(K)$ we have $q \geqslant k_*(Z_q^\circ(K))$, and Lemma 3.6(ii) shows that

$$c_3\sqrt{\frac{q}{n}}R(Z_q(K)) \leqslant w_q(Z_q(K)) \leqslant c_4\sqrt{\frac{q}{n}}R(Z_q(K)). \tag{3.16}$$

Now, from Lemma 3.2 we have that

$$w_q(Z_q(K)) = a_{n,q}\sqrt{\frac{q}{q+n}}I_q(K). \tag{3.17}$$

This proves (3.14). For the second assertion, we use (3.13) and the fact that $R(Z_2(K)) = L_K$ if K is isotropic. □

Remark. Let K be a convex body in $\mathbb{R}^n$, with volume 1 and center of mass at the origin. If $q \geqslant n$, one can check that $R(Z_q(K)) \simeq I_q(K) \simeq R(K)$.

Proposition 3.10. *There exists an absolute constant $c > 0$ with the following property: if K is a convex body of volume 1 in $\mathbb{R}^n$ which is ψ_α-body with constant b_α, then*

$$q_*(K) \geqslant c\frac{(k_*(Z_2^\circ(K)))^{\alpha/2}}{b_\alpha^\alpha}. \tag{3.18}$$

In particular, for every convex body K of volume 1 in $\mathbb{R}^n$ we have

$$q_*(K) \geqslant c\sqrt{k_*(Z_2^\circ(K))}. \tag{3.19}$$

Proof. Let $q_* := q_*(K)$. From Lemma 3.6(i), Lemma 3.2, Hölder's inequality and (3.6) we get

$$\begin{aligned} w(Z_{q_*}(K)) &\geqslant c_1 w_{q_*}(Z_{q_*}(K)) = c_1 a_{n,q_*}\sqrt{\frac{q_*}{n+q_*}}I_{q_*}(K) \\ &\geqslant c_1 a_{n,q_*}\sqrt{\frac{q_*}{n+q_*}}I_2(K) = c_1 a_{n,q_*}\sqrt{\frac{q_*}{n+q_*}}\,\sqrt{n}w_2(Z_2(K)). \end{aligned}$$

In other words,

$$w(Z_{q_*}(K)) \geqslant c_2\sqrt{q_*}w(Z_2(K)). \tag{3.20}$$

Since K is a ψ_α-body with constant b_α, we have that

$$R(Z_{q_*}(K)) \leqslant b_\alpha {q_*}^{1/\alpha} R(Z_2(K)). \tag{3.21}$$

Using the definition of q_*, Fact 3.4 and the inequalities (3.20) and (3.21), we write

$$\begin{aligned} q_* + 1 &\geqslant k_*(Z_{q_*}^\circ(K)) \geqslant c_3 n\left(\frac{w(Z_{q_*}(K))}{R(Z_{q_*}(K))}\right)^2 \\ &\geqslant c_3 n\frac{c_2^2 q_*}{b_\alpha^2 q_*^{2/\alpha}}\frac{w^2(Z_2(K))}{R^2(Z_2(K))} = c_5\frac{q_*^{1-2/\alpha}}{b_\alpha^2}k_*(Z_2^\circ(K)). \end{aligned}$$

So, we get

$$q_*(K) \geqslant c \frac{[k_*(Z_2^\circ(K))]^{\alpha/2}}{b_\alpha^\alpha}. \tag{3.22}$$

The second assertion follows from the fact that every convex body is a ψ_1-body with (an absolute) constant $c > 0$. □

Observe that K is isotropic if and only if $k_*(Z_2^\circ(K)) = n$. So, we get the following:

Corollary 3.11. *There exists an absolute constant $c > 0$ with the following property: if K is an isotropic convex body of volume 1 in $\mathbb{R}^n$ which is ψ_α-body with constant b_α, then*

$$q_*(K) \geqslant \frac{cn^{\alpha/2}}{b_\alpha^\alpha}. \tag{3.23}$$

In particular, for every isotropic convex body K in $\mathbb{R}^n$ we have that

$$q_*(K) \geqslant c\sqrt{n}. \tag{3.24}$$

4 Projections of L_q-centroid bodies

Let K be a convex body of volume 1 in $\mathbb{R}^n$. Let $F \in G_{n,k}$ be a k-dimensional subspace of $\mathbb{R}^n$ and let $q \geqslant 1$. We define

$$I_q(K,F) = \left(\int_K \|P_F(x)\|_2^q dx \right)^{1/q}, \tag{4.1}$$

where P_F denotes the orthogonal projection onto F, and

$$w_q(K,F) = \left(\int_{S_F} h_K^q(\theta)\, d\sigma_F(\theta) \right)^{1/q}, \tag{4.2}$$

where $S_F = S^{n-1} \cap F$ is the unit sphere of F. Observe that $w_q(K,F) = w_q(P_F(K))$. We also set

$$L(K,F) = \frac{I_2(K,F)}{\sqrt{k}} \tag{4.3}$$

and

$$L(K) = L(K,\mathbb{R}^n) = \frac{I_2(K)}{\sqrt{n}}. \tag{4.4}$$

The argument we used for the proof of Lemma 3.2 shows that

$$w_q(Z_q(K),F) = a_{k,q}\sqrt{\frac{q}{k+q}} I_q(K,F). \tag{4.5}$$

Choosing $q = 2$ and taking into account (4.3) we get

$$L^2(K,F) = \int_{S_F} h^2_{Z_2(K)}(\theta)d\sigma_F(\theta). \tag{4.6}$$

In particular, if K is isotropic then

$$L(K,F) = L(K) = L_K \tag{4.7}$$

for every F.

In the sequel, we fix a k-dimensional subspace F of $\mathbb{R}^n$ and denote by E the orthogonal subspace of F. For every $\phi \in S_F$ we define $E(\phi) = \{x \in \mathrm{span}\{E, \phi\} : \langle x, \phi\rangle \geqslant 0\}$.

Theorem 4.1 (K. Ball, see [4], [27]). *Let K a convex body of volume 1 in $\mathbb{R}^n$. For every $q \geqslant 0$ and $\phi \in F$, the function*

$$\phi \mapsto \|\phi\|_2^{1+\frac{q}{q+1}} \left(\int_{K\cap E(\phi)} |\langle x,\phi\rangle|^q dx\right)^{-\frac{1}{q+1}}$$

is a gauge function on F.

Note. In [4] and [27], Theorem 4.1 is stated and proved for the case where K is centrally symmetric. However, it was observed in [14] that the general case follows easily.

We denote by $B_q(K,F)$ the convex body in F whose gauge function is defined in Theorem 4.1. The volume of $B_q(K,F)$ is given by

$$|B_q(K,F)| = \omega_k \int_{S_F} \left(\int_{K\cap E(\phi)} |\langle x,\phi\rangle|^q dx\right)^{\frac{k}{q+1}} d\sigma_F(\phi). \tag{4.8}$$

To see this, express the volume of $B_q(K,F)$ in polar coordinates.

Lemma 4.2. *Let K be a convex body in $\mathbb{R}^n$. For every $q \geqslant 0$ and every $\theta \in S_F$, we have*

$$\int_K |\langle x,\theta\rangle|^q dx = k\omega_k \int_{S_F} |\langle \phi,\theta\rangle|^q \int_{K\cap E(\phi)} |\langle z,\phi\rangle|^{k+q-1} dz\, d\sigma_F(\phi). \tag{4.9}$$

Proof. For any continuous $f : \mathbb{R}^n \to \mathbb{R}$ we may write

$$\begin{aligned}
\int_K f(x)\,dx &= \int_E \int_F \chi_K(u+v) f(u+v)\,dv\,du \\
&= k\omega_k \int_E \int_{S_F} \int_0^\infty \chi_K(u+\rho\phi) f(u+\rho\phi)\rho^{k-1} d\rho\, d\sigma_F(\phi)\,du \\
&= k\omega_k \int_{S_F} \left(\int_E \int_0^\infty \chi_K(u+\rho\phi) f(u+\rho\phi)\rho^{k-1} d\rho\,du\right) d\sigma_F(\phi).
\end{aligned}$$

Observe that if $z = u + \rho\phi \in E(\phi)$ then $\rho = \langle z, \phi\rangle$. It follows that

$$\int_E \int_0^\infty \chi_K(u+\rho\phi) f(u+\rho\phi)\rho^{k-1} d\rho\, du = \int_{K\cap E(\phi)} f(z)\langle z,\phi\rangle^{k-1} dz. \tag{4.10}$$

In other words,

$$\int_K f(x)\, dx = k\omega_k \int_{S_F}\int_{K\cap E(\phi)} f(z)\langle z,\phi\rangle^{k-1} dz\, d\sigma_F(\phi). \tag{4.11}$$

Let $z \in K \cap E(\phi)$. Then $z = u + \langle \phi, z\rangle\phi$ for some $u \in E$, and hence, if $\theta \in F$ we have $\langle z, \theta\rangle = \langle \phi, \theta\rangle\langle z, \phi\rangle$. If we set $f_{\theta,q}(x) = |\langle x, \theta\rangle|^q$, then (4.11) becomes

$$\int_K |\langle x,\theta\rangle|^q dx = k\omega_k \int_{S_F} |\langle \phi,\theta\rangle|^q \int_{K\cap E(\phi)} \langle z,\phi\rangle^{k+q-1} dz\, d\sigma_F(\phi). \tag{4.12}$$

This completes the proof of (4.9). □

If we choose $q = 0$ in (4.9), we can express the volume of K in the following way:

$$|K| = k\omega_k \int_{S_F}\int_{K\cap E(\phi)} |\langle x,\phi\rangle|^{k-1} dx\, d\sigma_F(\phi). \tag{4.13}$$

Notation. If K is a convex body in $\mathbb{R}^n$, we set $\overline{K} = K/|K|^{1/n}$; this is the dilation of K which has volume 1.

Proposition 4.3. *Let K be a convex body of volume 1 in $\mathbb{R}^n$ and let $1 \leqslant k \leqslant n-1$. For every $F \in G_{n,k}$ and every $q \geqslant 1$ we have that*

$$P_F(Z_q(K)) = (k+q)^{1/q}\, |B_{k+q-1}(K,F)|^{1/k+1/q} Z_q(\overline{B}_{k+q-1}(K,F)). \tag{4.14}$$

Equivalently, for every $\theta \in F$,

$$\int_K |\langle x,\theta\rangle|^q dx = (k+q)\int_{B_{k+q-1}(K,F)} |\langle x,\theta\rangle|^q dx. \tag{4.15}$$

Proof. Let $\theta \in F$. Using polar coordinates on the right hand side of (4.15) and Lemma 4.2, we write

$$\begin{aligned}\int_{B_{k+q-1}(K,F)} |\langle x,\theta\rangle|^q dx &= \frac{k\omega_k}{k+q}\int_{S_F} |\langle\phi,\theta\rangle|^q \|\phi\|_{B_{k+q-1}(K,F)}^{-(k+q)}\, d\sigma_F(\phi)\\ &= \frac{k\omega_k}{k+q}\int_{S_F} |\langle\phi,\theta\rangle|^q \int_{K\cap E(\phi)} |\langle x,\phi\rangle|^{k+q-1} dx\, d\sigma_F(\phi)\\ &= \frac{1}{k+q}\int_K |\langle x,\theta\rangle|^q dx.\end{aligned}$$

This proves (4.15). Observe that $h_{P_F(Z_q(K))}(\theta) = h_{Z_q(K)}(\theta)$ for every $\theta \in F$. If we normalize the volume of $B_{k+q-1}(K,F)$, then (4.15) shows that

$$h_{P_F(Z_q(K))}(\theta) = (k+q)^{1/q}\, |B_{k+q-1}(K,F)|^{1/k+1/q} h_{Z_q(\overline{B}_{k+q-1}(K,F))}(\theta). \tag{4.16}$$

for every $\theta \in F$. This proves the Proposition. □

Notation. If a, b are positive integers, we define

$$B(b+1, a+1) := \int_0^1 s^a (1-s)^b ds = \frac{a!\, b!}{(a+b+1)!}. \tag{4.17}$$

One may easily check that

$$\left(\frac{b}{a}\right)^a \leqslant \binom{b}{a} \leqslant \left(\frac{eb}{a}\right)^a, \quad (0 < a < b) \tag{4.18}$$

and

$$b^a \leqslant \frac{(a+b)!}{b!} \leqslant (a+b)^a. \tag{4.19}$$

Let $n, k, q \in \mathbb{N}$, with $\max\{k, q\} < n$. We define

$$A_{n,k,q} := \left(\frac{B(n-k+1, k+q)^{\frac{k}{k+q}}}{B(n-k+1, k)} \right)^{\frac{k+q}{kq}}. \tag{4.20}$$

Lemma 4.4. *For every $n, k, q \in \mathbb{N}$, with $\max\{k, q\} < n$ we have that*

$$\frac{k^{\frac{1}{k}+\frac{1}{q}}}{(k+q)^{1/q}} \frac{n}{n+q} \leqslant A_{n,k,q} \leqslant e \frac{k^{\frac{1}{k}+\frac{1}{q}}}{(k+q)^{1/q}} \frac{k+q}{k}. \tag{4.21}$$

Proof. We first write $A_{n,k,q}$ in the form

$$A_{n,k,q} = \left(\frac{B(n-k+1, k+q)}{B(n-k+1, k)} \right)^{1/q} (B(n-k+1, k))^{-1/k}. \tag{4.22}$$

Using (4.17) we can write

$$\frac{B(n-k+1, k+q)}{B(n-k+1, k)} = \frac{k}{k+q} \frac{(k+q)!}{k!} \frac{n!}{(n+q)!} \tag{4.23}$$

and

$$(B(n-k+1, k))^{-1} = k \binom{n}{k}. \tag{4.24}$$

Using (4.19) into (4.23) we get

$$\frac{k}{k+q} \frac{k^q}{(n+q)^q} \leqslant \frac{B(n-k+1, k+q)}{B(n-k+1, k)} \leqslant \frac{k}{k+q} \frac{(k+q)^q}{n^q}. \tag{4.25}$$

Using (4.18) into (4.24) we get

$$k \left(\frac{n}{k}\right)^k \leqslant (B(n-k+1, k))^{-1} \leqslant k \left(\frac{en}{k}\right)^k. \tag{4.26}$$

Inserting (4.25) and (4.26) into (4.22) we get the Lemma. □

The following lemma is standard and goes back at least to Berwald [5] (see [9] and [27]).

Lemma 4.5. *Let C be a convex body in $\mathbb{R}^m$ and $0 \in \mathrm{int}(C)$. For every $\phi \in S^{m-1}$, set*

$$C_+(\phi) := \{x \in C : \langle x, \phi \rangle \geqslant 0\}. \tag{4.27}$$

If $s \leqslant r$ are non-negative integers, we have that

$$\left(\frac{\int_{C_+(\phi)} |\langle x, \phi \rangle|^r dx}{B(m, r+1)|C \cap \phi^\perp|} \right)^{1/(r+1)} \leqslant \left(\frac{\int_{C_+(\phi)} |\langle x, \phi \rangle|^s dx}{B(m, s+1)|C \cap \phi^\perp|} \right)^{1/(s+1)}. \tag{4.28}$$

Proposition 4.6. *Let K be a convex body of volume 1 in $\mathbb{R}^n$ and $0 \in \mathrm{int}(K)$. If $F \in G_{n,k}$ and $E = F^\perp$ then, for every integer $q \geqslant 1$,*

$$|B_{k+q-1}(K, F)|^{\frac{1}{k}+\frac{1}{q}} \leqslant \frac{e(k+q)}{k} \left(\frac{1}{k+q} \right)^{\frac{1}{q}} \frac{1}{|K \cap E|^{1/k}}. \tag{4.29}$$

Proof. By (4.8) we have that

$$|B_{k+q-1}(K, F)| = \omega_k \int_{S_F} \left(\int_{K \cap E(\phi)} |\langle x, \phi \rangle|^{k+q-1} dx \right)^{\frac{k}{k+q}} d\sigma_F(\phi). \tag{4.30}$$

Applying (4.28) with $C = K \cap \mathrm{span}\{E, \phi\}$, $m = n - k + 1$, $r = k + q - 1$ and $s = k - 1$, we get

$$\left(\frac{\int_{K \cap E(\phi)} |\langle x, \phi \rangle|^{k+q-1} dx}{B(n-k+1, k+q)|K \cap E|} \right)^{1/(k+q)} \leqslant \left(\frac{\int_{K \cap E(\phi)} |\langle x, \phi \rangle|^{k-1} dx}{B(n-k+1, k)|K \cap E|} \right)^{1/k} \tag{4.31}$$

or, equivalently,

$$\left(\int_{K \cap E(\phi)} |\langle x, \phi \rangle|^{k+q-1} dx \right)^{\frac{k}{k+q}} \leqslant \frac{A_{n,k,q}^{kq/(k+q)}}{(|K \cap E|)^{q/(k+q)}} \int_{K \cap E(\phi)} |\langle x, \phi \rangle|^{k-1} dx, \tag{4.32}$$

where $A_{n,k,q}$ is the constant defined by (4.20).

Going back to (4.30) and using (4.13) we get

$$\begin{aligned} |B_{k+q-1}(K, F)| &\leqslant \frac{A_{n,k,q}^{kq/(k+q)}}{|K \cap E|^{q/(k+q)}} \omega_k \int_{S_F} \int_{K \cap E(\phi)} |\langle x, \phi \rangle|^{k-1} dx \, d\sigma_F(\phi) \\ &= \frac{1}{k} \frac{A_{n,k,q}^{kq/(k+q)}}{|K \cap E|^{q/(k+q)}}. \end{aligned}$$

By Lemma 4.4 we conclude that

$$|B_{k+q-1}(K, F)|^{\frac{1}{k}+\frac{1}{q}} \leqslant \frac{e(k+q)}{k} \left(\frac{1}{k+q} \right)^{\frac{1}{q}} \frac{1}{|K \cap E|^{1/k}}, \tag{4.33}$$

as claimed. □

Lemma 4.7. *Let $f_1, f_2 : \mathbb{R}^k \to R$ be integrable functions with compact support such that $\int_{\mathbb{R}^k} f_1(x)dx = \int_{\mathbb{R}^k} f_2(x)dx$ and, for every $s > 0$, $\int_{sB_2^k} f_1(x)dx \leqslant \int_{sB_2^k} f_2(x)dx$. Then, for every $p > 0$,*

$$\int_{\mathbb{R}^k} \|x\|_2^p f_1(x)dx \geqslant \int_{\mathbb{R}^k} \|x\|_2^p f_2(x)dx. \tag{4.34}$$

Proof. We write

$$\begin{aligned}\int_{\mathbb{R}^k} \|x\|_2^p f_i(x)\,dx &= \int_{\mathbb{R}^k} \int_0^{\|x\|_2} ps^{p-1} f_i(x)\,ds\,dx \\ &= \int_0^{\infty} ps^{p-1} \int_{(sB_2^k)^c} f_i(x)\,dx\,ds,\end{aligned}$$

and observe that $\int_{(sB_2^k)^c} f_1(x)dx \geqslant \int_{(sB_2^k)^c} f_2(x)dx$ for every $s \geqslant 0$. □

Proposition 4.8. *Let K be a convex body in $\mathbb{R}^n$, with volume 1 and center of mass at the origin. Let $F \in G_{n,k}$ and $E := F^{\perp}$ Then*

$$\frac{1}{|K \cap E|^{1/k}} \leqslant cL(K,F), \tag{4.35}$$

where $c > 0$ is an absolute constant.

Proof. Let $M := \sup_{x\in F} |K \cap (E + x)|$, $f_1(x) := |K \cap (E + x)|$ and $f_2(x) := M\chi_{\omega_k^{-1/k} M^{-1/k} B_F}(x)$, where $B_F = B_2^n \cap F$. Then,

$$\int_F f_1(x)\,dx = 1 = \int_F f_2(x)\,dx, \tag{4.36}$$

and, from the fact that f_2 is equal to M on a ball centered at the origin and equal to zero elsewhere, we easily check that

$$\int_{sB_F} f_1(x)\,dx \leqslant \int_{sB_F} f_2(x)\,dx \tag{4.37}$$

for every $s > 0$. Lemma 4.7 shows that

$$\int_F \|x\|_2^2 f_1(x)\,dx \geqslant \int_F \|x\|_2^2 f_2(x)\,dx = \frac{k}{k+2}\omega_k^{-2/k} M^{-2/k} = I_2^2(\overline{B}_F)M^{-2/k}. \tag{4.38}$$

Observe that

$$\int_F \|x\|_2^2 f_1(x)\,dx = \int_K \|P_F x\|_2^2 dx = I_2^2(K,F) = k(L(K,F))^2. \tag{4.39}$$

A result of Fradelizi (see [16]) shows that $M \leqslant e^k |K \cap E|$. This proves (4.35). □

Proposition 4.9. *Let K be a convex body in $\mathbb{R}^n$, with volume 1 and center of mass at the origin. If $F \in G_{n,k}$ and $E = F^{\perp}$ then, for every $q \in \mathbb{N}$ we have that*

$$P_F(Z_q(K)) \subseteq \frac{c(k+q)}{k} L(K,F) Z_q(\overline{B}_{k+q-1}(K,F)) \tag{4.40}$$

where $c > 0$ is an absolute constant.

Proof. We start from Proposition 4.3 and use Propositions 4.6 and 4.8 to estimate the quantity $|B_{k+q-1}(K,F)|^{1/k+1/q}$ which appears in (4.14). □

Proposition 4.10. *Let K be a convex body in $\mathbb{R}^n$, with volume 1 and center of mass at the origin. For every k-dimensional subspace F of $\mathbb{R}^n$ and every integer $q \geqslant 1$ there exists $\theta \in S_F$ such that*

$$h_{Z_q(K)}(\theta) \leqslant c\sqrt{k}\,\frac{k+q}{k} L(K,F), \tag{4.41}$$

where $c > 0$ is an absolute constant.

Proof. By Proposition 4.9, taking volumes in (4.40), we have that

$$|P_F(Z_q(K))|^{1/k} \leqslant \frac{4c(k+q)}{k} L(K,F) |Z_q(\overline{B}_{k+q-1}(K,F))|^{1/k}. \tag{4.42}$$

Recall that

$$\begin{aligned} Z_q(\overline{B}_{k+q-1}(K,F)) &\subseteq \mathrm{conv}\{\overline{B}_{k+q-1}(K,F), -\overline{B}_{k+q-1}(K,F)\} \\ &\subseteq \overline{B}_{k+q-1}(K,F) - \overline{B}_{k+q-1}(K,F). \end{aligned}$$

By the Rogers–Shephard inequality (see [35]) we have that

$$|Z_q(\overline{B}_{k+q-1}(K,F))|^{1/k} \leqslant 4. \tag{4.43}$$

Therefore,

$$|P_F(Z_q(K))|^{1/k} \leqslant \frac{4c(k+q)}{k} L(K,F). \tag{4.44}$$

Assume that

$$\rho(B_2^n \cap F) \subseteq P_F(Z_q(K)) \tag{4.45}$$

for some $\rho > 0$. The Proposition will be proved if we show that

$$\rho \leqslant c\sqrt{k}\,\frac{k+q}{k} L(K,F). \tag{4.46}$$

From (4.44) and (4.45) we get

$$\rho\omega_k^{1/k} \leqslant \frac{4c(k+q)}{k} L(K,F). \tag{4.47}$$

Since $\omega_k^{1/k} \simeq 1/\sqrt{k}$ we get (4.41). □

Corollary 4.11. *Let K be a convex body in $\mathbb{R}^n$, with volume 1 and center of mass at the origin. For every integer $q \geqslant 1$ and every $F \in G_{n,q}$ there exists $\theta \in S_F$ such that*

$$h_{Z_q(K)}(\theta) \leqslant c\sqrt{q}\, L(K,F), \tag{4.48}$$

where $c > 0$ is an absolute constant.

5 Proof of the main result

We are now ready to give the proof of Theorem 2.3. The precise formulation of our result in the isotropic case is the following.

Theorem 5.1. *There exists an absolute constant $c > 0$ with the following property: if K is an isotropic convex body in $\mathbb{R}^n$, then*

$$I_q(K) \leqslant cI_2(K) \tag{5.1}$$

for every $q \leqslant q_*(K)$.

Proof. Set $q_* = q_*(K)$. By the definition of $q_*(K)$ and $k_*(Z^\circ_{q_*}(K))$ we have $k_*(Z^\circ_{q_*}(K)) \geqslant q_*$, and hence, there exists a q_*-dimensional subspace F of $\mathbb{R}^n$ such that

$$h_{Z_{q_*}(K)}(\theta) \geqslant \tfrac{1}{2}w(Z_{q_*}(K)) \tag{5.2}$$

for every $\theta \in S_F$.

On the other hand, Corollary 4.11 shows that there exists $\theta_0 \in S_F$ such that

$$h_{Z_{q_*}(K)}(\theta_0) \leqslant c_1\sqrt{q_*}L(K,F) = c_1\sqrt{q_*}L_K, \tag{5.3}$$

where $c_1 > 0$ is an absolute constant (here, we are using the fact that K is isotropic; we have $L(K,F) = L_K$ for every subspace F of $\mathbb{R}^n$). It follows that

$$w(Z_{q_*}(K)) \leqslant 2c_1\sqrt{q_*}L_K. \tag{5.4}$$

Since $q_* \leqslant k_*(Z^\circ_{q_*}(K))$, from Lemma 3.5 and Lemma 3.2 we have

$$w(Z_{q_*}(K)) \geqslant c_2 w_{q_*}(Z_{q_*}(K)) \geqslant c_3\sqrt{\frac{q_*}{n}}I_{q_*}(K). \tag{5.5}$$

Combining (5.4) and (5.5) we see that

$$I_{q_*}(K) \leqslant c\sqrt{n}L_K, \tag{5.6}$$

for some absolute constant $c > 0$. Since $\sqrt{n}L_K = I_2(K)$, the result follows. □

Proof of Theorem 2.2. We have assumed that K is isotropic. Then, Corollary 3.11 shows that $q_*(K) \geqslant c\sqrt{n}$, where $c > 0$ is an absolute constant. Then, Theorem 2.2 is an immediate consequence of Theorem 5.1. □

In fact, the method which has been developed in the previous Sections provides a similar result for an arbitrary convex body that has its center of mass at the origin:

Theorem 5.2. *Let K be a convex body in $\mathbb{R}^n$, with volume 1 and center of mass at the origin. If $q_* = q_*(K)$, then*

$$I_{q_*}(K) \leqslant cI_2(K), \tag{5.7}$$

where $c > 0$ is an absolute constant.

For the proof of Theorem 5.2 we need one more Lemma.

Lemma 5.3. *There exists a constant $c \in (0,1)$ with the following property: if C is a symmetric convex body in $\mathbb{R}^n$ and if $m \leqslant k_*(C^\circ) \leqslant cn$, then*

$$w(C) \leqslant 2\int_B \int_{S_F} h_C(\theta)\, d\sigma(\theta)\, d\mu_{n,m}(F), \tag{5.8}$$

where

$$B = \left\{F \in G_{n,m} : \tfrac{1}{2}w(C) \leqslant h_C(\theta) \leqslant 2w(C) \text{ for all } \theta \in S_F\right\}. \tag{5.9}$$

Proof. Since $m \leqslant k_*(C^\circ)$, we have that $\mu_{n,m}(B^c) \leqslant \frac{m}{n+m}$, where $B^c = G_{n,m} \setminus B$. Then, using the fact that

$$w_2(C) \leqslant c_1 w(C) \tag{5.10}$$

which can be easily checked from Lemma 3.6, we can write

$$\begin{aligned}
w(C) &= \int_{G_{n,m}} \int_{S_F} h_K(\theta)\, d\sigma_F(\theta)\, d\mu_{n,m}(F) \\
&= \int_B \int_{S_F} h_C(\theta)\, d\sigma_F(\theta)\, d\mu_{n,m}(F) + \int_{B^c} \int_{S_F} h_C(\theta)\, d\sigma_F(\theta)\, d\mu_{n,m}(F) \\
&\leqslant \int_B \int_{S_F} h_C(\theta)\, d\sigma_F(\theta)\, d\mu_{n,m}(F) \\
&\quad + \big(\mu(B^c)\big)^{1/2} \left(\int_{G_{n,m}} \left(\int_{S_F} h_C(\theta)\, d\sigma_F(\theta)\right)^2 d\mu_{n,m}(F)\right)^{1/2} \\
&\leqslant \int_B \int_{S_F} h_C(\theta)\, d\sigma_F(\theta)\, d\mu_{n,m}(F) \\
&\quad + \big(\mu(B^c)\big)^{1/2} \left(\int_{G_{n,m}} \int_{S_F} h_C^2(\theta)\, d\sigma_F(\theta)\, d\mu_{n,m}(F)\right)^{1/2} \\
&\leqslant \int_B \int_{S_F} h_C(\theta)\, d\sigma_F(\theta)\, d\mu_{n,m}(F) + \sqrt{\frac{m}{n+m}}\, w_2(C) \\
&\leqslant \int_B \int_{S_F} h_C(\theta)\, d\sigma_F(\theta)\, d\mu_{n,m}(F) + c_1\sqrt{\frac{m}{n+m}}\, w(C) \\
&\leqslant \int_B \int_{S_F} h_C(\theta)\, d\sigma_F(\theta)\, d\mu_{n,m}(F) + \tfrac{1}{2}w(C),
\end{aligned}$$

provided that $c \in (0,1)$ is chosen small enough. $\square$

Proof of Theorem 5.2. We define

$$q = \min\{q_*, \lfloor cn \rfloor\} \tag{5.11}$$

where $c \in (0,1)$ is the constant from Lemma 5.3. By Lemma 3.2 and Lemma 3.6 we get

$$I_q(K) = a_{n,q}^{-1}\sqrt{\frac{q+n}{q}}\, w_q(Z_q(K)) \leqslant c_1 a_{n,q}^{-1}\sqrt{\frac{q+n}{q}}\, w(Z_q(K)). \tag{5.12}$$

Set

$$B = \left\{F \in G_{n,q} : \tfrac{1}{2} w(Z_q(K)) \leqslant h_{Z_q(K)}(\theta) \leqslant 2w(Z_q(K)) \text{ for all } \theta \in S_F\right\}. \tag{5.13}$$

From Lemma 5.3 we have that

$$w(Z_q(K)) \leqslant 2\int_B \int_{S_F} h_{Z_q(K)}(\theta)\, d\sigma(\theta)\, d\mu_{n,q}(F). \tag{5.14}$$

Now, Corollary 4.11 and the definition of B show that, for every $F \in B$, *there exists* $\theta_0 \in S_F$ such that

$$w(Z_q(K)) \leqslant 2h_{Z_q(K)}(\theta_0) \leqslant 2c_2\sqrt{q}\, L(K,F). \tag{5.15}$$

Using again the definition of B, we now see that for every $F \in B$ and *for every* $\theta \in S_F$ we have that

$$h_{Z_q(K)}(\theta) \leqslant 2w(Z_q(K)) \leqslant 4c_2\sqrt{q}\, L(K,F). \tag{5.16}$$

In view of (4.6) this means that, for every $F \in B$ and for every $\theta \in S_F$,

$$h_{Z_q(K)}(\theta) \leqslant 2w(Z_q(K)) \leqslant 4c_2\sqrt{q}\left(\int_{S_F} h^2_{Z_2(K)}(\phi) d\sigma_F(\phi)\right)^{1/2}. \tag{5.17}$$

Going back to (5.14) we may write

$$\begin{aligned} w(Z_q(K)) &\leqslant 8c_2\sqrt{q}\int_B\int_{S_F}\left(\int_{S_F} h^2_{Z_2(K)}(\phi)d\sigma_F(\phi)\right)^{1/2} d\sigma_F(\theta)\, d\mu_{n,q}(F) \\ &= 8c_2\sqrt{q}\int_B\left(\int_{S_F} h^2_{Z_2(K)}(\phi)d\sigma_F(\phi)\right)^{1/2} d\mu_{n,q}(F) \\ &\leqslant 8c_2\sqrt{q}\left(\int_{G_{n,q}}\int_{S_F} h^2_{Z_2(K)}(\phi)d\sigma_F(\phi)\, d\mu_{n,q}(F)\right)^{1/2} \\ &= 8c_2\sqrt{q}L(K). \end{aligned}$$

Then, (5.12) becomes

$$I_q(K) \leqslant c_1 a_{n,q}^{-1}\sqrt{\frac{q+n}{q}} \cdot (8c_2\sqrt{q}L(K)) \leqslant c_3\sqrt{n}\,L(K). \tag{5.18}$$

Since $\sqrt{n}L(K) = I_2(K)$ by definition (see (4.4)), we finally get

$$I_q(K) \leqslant c_3 I_2(K). \tag{5.19}$$

From Lemma 2.1 we know that

$$I_s(K) \leqslant c_4 \frac{s}{p} I_p(K) \tag{5.20}$$

for all $s \geqslant p \geqslant 1$, where $c_4 > 0$ is an absolute constant, and hence, we can compare $I_{q_*}(K)$ with $I_q(K)$. This completes the proof. □

Corollary 5.4. *Let K be an isotropic convex body in $\mathbb{R}^n$, which is ψ_a-body with constant b_α. Then,*

$$I_q(K) \leqslant c \max\{b_\alpha q^{1/\alpha}, \sqrt{n}\} L_K \tag{5.21}$$

for every $2 \leqslant q \leqslant n$, where $c > 0$ is an absolute constant. In particular, for every isotropic convex body K in $\mathbb{R}^n$ we have that

$$I_q(K) \leqslant c_1 \max\{q, \sqrt{n}\} L_K \tag{5.22}$$

for every $2 \leqslant q \leqslant n$, where $c_1 > 0$ is an absolute constant.

Proof. Direct consequence of Theorem 5.1 and Lemma 3.9. □

It is interesting to note that the Euclidean ball and the ℓ_1^n-ball B_1^n are the extremal bodies in Theorem 5.2, in the following sense:

Proposition 5.5. *Let K be a convex body of volume 1 in $\mathbb{R}^n$. For every $0 < p < q < \infty$ we have that*

$$\frac{I_q(K)}{I_p(K)} \geqslant \frac{I_q(\overline{B_2^n})}{I_p(\overline{B_2^n})}. \tag{5.23}$$

Proof. We follow an argument of Bobkov–Koldobsky from [6]. Let $0 < p < q < \infty$. A simple computation shows that

$$I_p^p(\overline{B_2^n}) = n\omega_n \int_0^{\omega_n^{-1/n}} r^{n+p-1} dr = \frac{n}{n+p}\omega_n^{-p/n}. \tag{5.24}$$

Therefore,

$$\frac{I_q(\overline{B_2^n})}{I_p(\overline{B_2^n})} = \frac{(\frac{n}{n+q})^{1/q}}{(\frac{n}{n+p})^{1/p}}. \tag{5.25}$$

For every $q > -n$ we have that

$$I_q^q(K) = \omega_n \int_0^\infty r^{n+q-1} \sigma\left(\frac{1}{r}K\right) dr. \tag{5.26}$$

The function $g(r) = \omega_n \sigma\left(\frac{1}{r}K\right)$ is non-increasing on $(0,\infty)$ and can be assumed absolutely continuous. So, we can write

$$g(r) = n \int_r^\infty \frac{\rho(s)}{s^n} ds, \quad (r > 0) \tag{5.27}$$

for some non-negative function ρ on $(0,\infty)$. Then,

$$1 = \int_0^\infty r^{n-1} g(r)\, dr = n \int_0^\infty \int_{0<r<s} r^{n-1} \frac{\rho(s)}{s^n} dr\, ds = \int_0^\infty \rho(s)\, ds. \tag{5.28}$$

Hence, ρ represents a probability density of a positive random variable, say, ξ. We now write

$$I_q^q(K) = \int_0^\infty r^{q+n-1} g(r)\, dr = \frac{n}{n+q} \int_0^\infty s^q \rho(s)\, ds = \frac{n}{n+q}\mathbb{E}(\xi^q). \tag{5.29}$$

Applying Hölder's inequality for $0 < p \leqslant q \leqslant \infty$, we see that

$$\left(\mathbb{E}(\xi^q)\right)^{1/q} \geqslant \left(\mathbb{E}(\xi^p)\right)^{1/p}. \tag{5.30}$$

So,

$$\frac{I_q(K)}{I_p(K)} = \frac{\left(\frac{n}{n+q}\mathbb{E}(\xi^q)\right)^{1/q}}{\left(\frac{n}{n+p}\mathbb{E}(\xi^p)\right)^{1/p}} \geqslant \frac{\left(\frac{n}{n+q}\right)^{1/q}}{\left(\frac{n}{n+p}\right)^{1/p}} = \frac{I_q(\overline{B_2^n})}{I_p(\overline{B_2^n})}, \tag{5.31}$$

as claimed. □

We now pass to the ℓ_1^n-ball; the results of [39] show that

$$I_q(\overline{B_1^n}) \simeq \max\{q, \sqrt{n}\} L_{\overline{B_1^n}} \tag{5.32}$$

for every $2 \leqslant q \leqslant n$. We will prove something more general:

Lemma 5.6. *Let K be an isotropic convex body in $\mathbb{R}^n$. Then, for every $1 \leqslant q \leqslant n$ we have*

$$I_q(K) \geqslant \frac{cq}{n} R(K), \tag{5.33}$$

where $c > 0$ is an absolute constant.

Proof. From the Remark after Lemma 3.9 we know that for every convex body K of volume 1 with center of mass at the origin, $R(K) \leqslant c_1 I_n(K)$, where $c_1 > 0$ is an absolute constant. Also, Lemma 2.1 shows that, for every $p, q \geqslant 1$,

$$I_{pq}(K) \leqslant c_2 p I_q(K) \tag{5.34}$$

where $c_2 > 0$ is an absolute constant.

Let $1 \leqslant q \leqslant n$. Then,

$$R(K) \leqslant c_1 I_n(K) \leqslant c_1 c_2 \frac{n}{q} I_q(K). \tag{5.35}$$

This proves the Lemma, with $c := \frac{1}{c_1 c_2}$. □

Remark. Since $R(\overline{B_1^n}) \simeq n L_{\overline{B_1^n}}$, Lemma 5.6 and (5.22) prove (5.32).

Corollary 5.7. *There exists an absolute constant $c > 0$ such that for every isotropic convex body K in $\mathbb{R}^n$ and for every $2 \leqslant q \leqslant \infty$,*

$$\frac{I_q(\overline{B_2^n})}{I_2(\overline{B_2^n})} \leqslant \frac{I_q(K)}{I_2(K)} \leqslant c \frac{I_q(\overline{B_1^n})}{I_2(\overline{B_1^n})}. \tag{5.36}$$

Proof of Theorem 1.4. Let K be an isotropic convex body in $\mathbb{R}^n$. If $T \in SL(n)$ is positive definite, then

$$I_2^2(T(K)) = \int_{T(K)} \|x\|_2^2 dx = \int_K \langle x, (T^*T)(x)\rangle \, dx = \mathrm{tr}(T^*T)\, L_K^2, \tag{5.37}$$

by the isotropicity of K. Since $I_2(K) = \sqrt{n} w_2(Z_2(T(K)))$ and $\mathrm{tr}(T^*T) = \|T\|_{HS}^2$, we get

$$w_2(Z_2(T(K))) = \frac{\|T\|_{HS}}{\sqrt{n}} L_K. \tag{5.38}$$

On the other hand,

$$R(Z_2(T(K))) = R(T(Z_2(K))) = R(T(L_K B_2^n)) = L_K R(T(B_2^n)) = L_K \lambda_1(T), \tag{5.39}$$

where $\lambda_1(T)$ is the largest eigenvalue of T. It follows that

$$k_*(Z_2^\circ(TK)) \simeq n \left(\frac{w(Z_2(T(K)))}{R(Z_2(T(K)))}\right)^2 \simeq \left(\frac{\|T\|_{HS}}{\lambda_1(T)}\right)^2. \tag{5.40}$$

From Proposition 3.10 we know that $q_*(T(K)) \geqslant c\sqrt{k_*(Z_2^\circ(K))}$, and hence, Theorem 5.2 and the reduction scheme of Section 2 show that

$$\mathrm{Prob}\left(\{x \in K : \|x\|_2 \geqslant c I_2(K) t\}\right) \leqslant \exp\left(-\frac{\|T\|_{HS}}{\lambda_1(T)} t\right) \tag{5.41}$$

for every $t \geqslant 1$, where $c > 0$ is an absolute constant, which is the assertion of Theorem 1.4.

6 Volume of L_q-centroid bodies

The L_q-affine isoperimetric inequality of Lutwak, Yang and Zhang (see Theorem 3.1) can be written in the following form.

Proposition 6.1. *Let K be a convex body in $\mathbb{R}^n$, with volume 1 and center of mass at the origin. Then,*

$$|Z_q(K)|^{1/n} \geqslant |Z_q(\overline{B_2^n})|^{1/n} \geqslant c\sqrt{q/n} \tag{6.1}$$

for every $1 \leqslant q \leqslant n$, where $c > 0$ is an absolute constant. □

Our goal in this Section is to show that the reverse inequality holds true (up to the isotropic constant).

Theorem 6.2. *Let K be a convex body in $\mathbb{R}^n$, with volume 1 and center of mass at the origin. For every $2 \leqslant q \leqslant n$ we have that*

$$|Z_q(K)|^{1/n} \leqslant c\sqrt{q/n}\, L_K, \tag{6.2}$$

where $c > 0$ is an absolute constant.

For the proof we will use the Aleksandrov–Fenchel inequalities for the quermassintegrals of a convex body C. From the classical Steiner's formula we know that

$$|C + tB_2^n| = \sum_{k=0}^{n} \binom{n}{k} W_{[k]}(C) t^k \tag{6.3}$$

for all $t > 0$, where $W_{[k]}(C)$ is the k-th quermassintegral of C; $W_{[k]}(C)$ is the mixed volume $V_{n-k}(C) = V(C; n-k, B_2^n; k)$.

The Aleksandrov–Fenchel inequality implies the log-concavity of the sequence $(W_{[0]}(C), \ldots, W_{[n]}(C))$. In other words,

$$W_{[j]}^{k-i}(C) \geqslant W_{[i]}^{k-j}(C) W_{[k]}^{j-i}(C), \quad (0 \leqslant i < j < k \leqslant n). \tag{6.4}$$

Choosing $k = n$ we see that

$$\left(\frac{W_{[i]}(C)}{\omega_n}\right)^{1/(n-i)} \leqslant \left(\frac{W_{[j]}(C)}{\omega_n}\right)^{1/(n-j)}, \tag{6.5}$$

for all $1 \leqslant i < j < n$.

We will also use Kubota's integral formula which connects the i-th quermassintegral with the average of the volumes of the $(n-i)$-dimensional projections of C:

$$W_{[i]}(C) = \frac{\omega_n}{\omega_{n-i}} \int_{G_{n,n-i}} |P_F(C)| d\mu_{n,n-i}(F), \quad (1 \leqslant i \leqslant n-1). \tag{6.6}$$

Proof of Theorem 6.2. We may assume that K is isotropic. It is enough to prove (6.2) for $q \in \mathbb{N}$ and $1 \leqslant q \leqslant n-1$.

Taking $k = q$ in (4.45) we see that

$$|P_F(Z_q(K))|^{1/q} \leqslant c_1 L_K, \tag{6.7}$$

where $c_1 > 0$ is an absolute constant. Applying (6.6) we get

$$W_{[n-q]}(Z_q(K)) \leqslant \frac{\omega_n}{\omega_q}(c_1 L_K)^q. \tag{6.8}$$

Now, we apply (6.5) for $C = Z_q(K)$ with $j = n - q$ and $i = 0$; this gives

$$W_{[n-q]}^n(Z_q(K)) \geqslant |Z_q(K)|^q \omega_n^{n-q} \tag{6.9}$$

or, equivalently,

$$W_{[n-q]}^{1/q}(Z_q(K)) \geqslant |Z_q(K)|^{1/n} \omega_n^{1/q-1/n}. \tag{6.10}$$

Combining (6.8) and (6.10) we get

$$|Z_q(K)|^{1/n} \leqslant \frac{\omega_n^{1/n}}{\omega_q^{1/q}} c L_K. \tag{6.11}$$

Since $\omega_k^{1/k} \simeq 1/\sqrt{k}$, the result follows. □

7 Random points in isotropic symmetric convex bodies

For the proof of Theorem 1.6 we follow the argument of [18] which incorporates the concentration estimate of Theorem 1.1 into Rudelson's approach to the problem. The main lemma in [36] is the following.

Theorem 7.1 (Rudelson). *Let $x_1, \ldots, x_N$ be vectors in $\mathbb{R}^n$ and let $\varepsilon_1, \ldots, \varepsilon_N$ be independent Bernoulli random variables which take the values ± 1 with probability $1/2$. Then, for all $p \geqslant 1$,*

$$\left(\mathbb{E} \left\| \sum_{i=1}^N \varepsilon_i x_i \otimes x_i \right\|^p \right)^{1/p} \leqslant c\sqrt{p + \log n} \cdot \max_{i \leqslant N} \|x_i\|_2 \cdot \left\| \sum_{i=1}^N x_i \otimes x_i \right\|^{1/2}, \tag{7.1}$$

where $c > 0$ is an absolute constant. □

Proof of Theorem 1.6. Let $\varepsilon \in (0,1)$ and let $p \geqslant 1$. We first estimate the expectation of $\max_{i \leqslant N} \|x_i\|_2^{2p}$, where $x_1, \ldots, x_N$ are independent random points uniformly distributed in K.

Lemma 7.2. *There exists $c > 0$ such that for every isotropic convex body K in $\mathbb{R}^n$, for every $N \in \mathbb{N}$ and every $p \geqslant 1$,*

$$\left(\mathbb{E} \max_{i\leqslant N} \|x_i\|_2^p\right)^{1/p} \leqslant cL_K \max\{\sqrt{n}, p, \log N\}. \tag{7.2}$$

Proof. From Theorem 1.1 we have

$$\mathrm{Prob}(x \in K : \|x\|_2 \geqslant cqL_K) \leqslant \exp(-q) \tag{7.3}$$

for every $q \geqslant \sqrt{n}$, where $c > 0$ is an absolute constant. We set $A := \max\{p, \sqrt{n}, \log N\}$. Since $A \geqslant \sqrt{n}$, we may write

$$\begin{aligned}
\mathbb{E} \max_{i\leqslant N} \|x_i\|_2^p &= \int_0^\infty pt^{p-1}\mathrm{Prob}(\max_{i\leqslant N} \|x_i\|_2 \geqslant t)\, dt \\
&\leqslant c^p L_K^p \int_0^A pt^{p-1}dt + pc^p L_K^p N \int_A^\infty t^{p-1}\mathrm{Prob}(\|x\|_2 \geqslant ctL_K)\, dt \\
&\leqslant c^p L_K^p A^p + pc^p L_K^p N \int_A^\infty t^{p-1}e^{-t}dt \\
&\leqslant c^p L_K^p A^p + pc^p L_K^p N e^{-A+1}A^p \\
&\leqslant c^p L_K^p A^p(1 + epNe^{-A}) \\
&\leqslant c^p L_K^p A^p(1 + ep)
\end{aligned}$$

where we have used the fact that

$$\int_A^\infty t^{p-1}e^{-t}dt \leqslant e^{-A+1}A^p \tag{7.4}$$

for all $A \geqslant p \geqslant 1$. □

Following Rudelson's argument (see also [18], page 10) we see that if $x_1', \ldots, x_N'$ are independent random points from K which are chosen independently from the x_i's, then

$$S^p := \mathbb{E}\left\|I - \frac{1}{NL_K^2}\sum_{i=1}^N x_i \otimes x_i\right\|^p \leqslant (4c)^p \frac{(p + \log n)^{p/2}}{N^{p/2}L_K^p}\left(\mathbb{E} \max_{i\leqslant N} \|x_i\|_2^{2p}\right)^{1/2} \sqrt{S^p + 1}. \tag{7.5}$$

If we choose $p = \log n$, Lemma 7.2 and (7.5) show that

$$S^p \leqslant \left(\frac{c_1(\log n)\max\{n, (\log N)^2\}}{N}\right)^{p/2} \sqrt{S^p + 1}. \tag{7.6}$$

From this inequality we see that if $N \geqslant c(\varepsilon)n \log n$ then

$$\left(\frac{c_1(\log n)\max\{n, (\log N)^2\}}{N}\right)^{p/2} < \frac{\varepsilon^{p+1}}{2}, \tag{7.7}$$

and hence,

$$\mathbb{E}\left\| I - \frac{1}{NL_K^2}\sum_{i=1}^N x_i \otimes x_i \right\|^p = S^p < \varepsilon^{p+1}. \tag{7.8}$$

An application of Markov's inequality shows that

$$\text{Prob}\left(\left\| I - \frac{1}{NL_K^2}\sum_{i=1}^N x_i \otimes x_i \right\| > \varepsilon \right) < \varepsilon, \tag{7.9}$$

which is exactly the assertion of Theorem 1.6. □

8 Concluding Remarks

All the main results of this paper remain valid if we replace Lebesgue measure on an isotropic convex body by an arbitrary isotropic log-concave measure. In our discussion, the fact that K is a convex body was only used through the log-concavity of the function $t \to |\{x \in K : |\langle x, \theta\rangle| = t\}|$. Also our assumption that K has centre of mass at the origin was needed in order to use Fradelizi's Theorem which is also valid for any log-concave probability mesure. One way to extend our results to the case of a log-concave probability measure in $\mathbb{R}^n$ is to introduce the relevant parameters and follow the proofs of the previous Sections:

Let μ be a log-concave probability measure in $\mathbb{R}^n$. We say that μ has its center of mass at the origin if $\int_{\mathbb{R}^n} \langle x, \theta\rangle \, d\mu(x) = 0$ for all $\theta \in S^{n-1}$. For $q \geqslant 1$ we define $I_q(\mu) := \left(\int_{\mathbb{R}^n} \|x\|_2^q d\mu(x)\right)^{1/q}$ and we consider the symmetric convex body $Z_q(\mu)$ in $\mathbb{R}^n$ which has support function $h_{Z_q(\mu)}(\theta) := \left(\int_{\mathbb{R}^n} |\langle x, \theta\rangle|^q d\mu(x)\right)^{1/q}$.

Next, we define

$$q_*(\mu) = \max\{q \in \mathbb{N} : k_*(Z_q^\circ(\mu)) \geqslant q\}. \tag{8.1}$$

Then, one can prove the following analogue of Theorem 5.2:

Theorem 8.1. *Let μ be a log-concave probability measure in $\mathbb{R}^n$ with center of mass at the origin. Then, for every $q \leqslant q_*(\mu)$,*

$$I_q(\mu) \leqslant cI_2(\mu) \tag{8.2}$$

where $c > 0$ is an absolute constant.

The proof of Theorem 8.1 is similar to the proof of Theorem 5.2; only a few straightforward modifications are needed.

Let μ be a log-concave probability measure in $\mathbb{R}^n$. We say that μ is isotropic if $Z_2(\mu)$ is a multiple of the Euclidean ball. An inspection of the proofs in Section 3 makes it clear that Proposition 3.10 and Corollary 3.11 remain true in the "log-concave" case. This implies immediately a reformulation of Theorem 2.3 for log-concave measures.

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Added in Proofs: B. Klartag has recently proved that for every convex body K in $\mathbb{R}^n$ and for every $\varepsilon > 0$ there exists a second convex body T in $\mathbb{R}^n$ whose Banach-Mazur distance from K is bounded by $1+\varepsilon$ and and its isotropic constant satisfies $L_T \leqslant C/\sqrt{\varepsilon}$. This almost isometric answer to the slicing problem, combined with Theorem 1.1 of our paper, leads to the estimate $L_K \leqslant c\sqrt[4]{n}$ for every convex body K. Klartag's work will appear in this Journal.