Permutation invariant balls and other stories

Probability in Asymptotic Geometry, College Station

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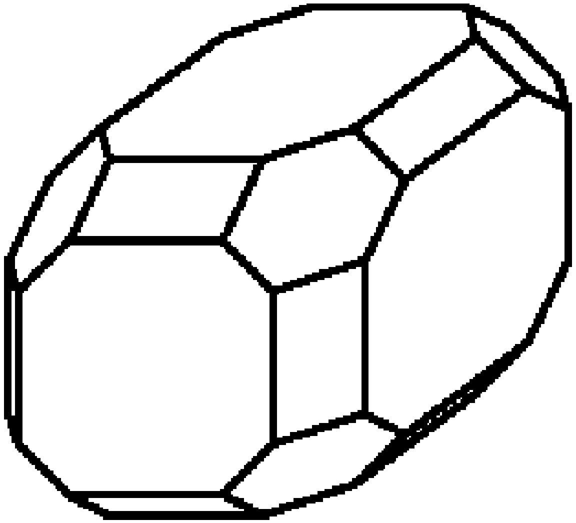
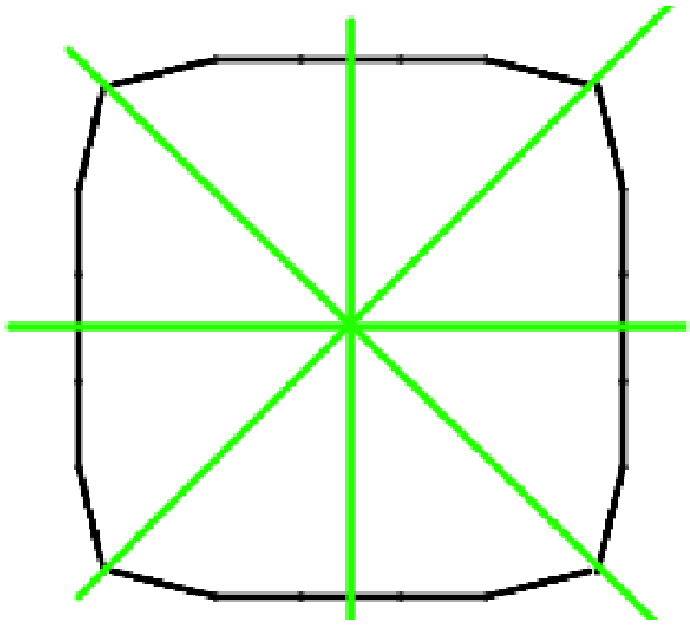
Definition

Permutation invariant ball: a convex set K with $AK = K$ for K being any permutation matrix (unit ball of a symmetric norm). I additionally assume unconditionality and volume 1. Examples: ℓ_p^n, Orlicz balls, many more.





Notation

- We work in $\mathbb{R}^n$, n denotes the dimension
- K — permutation invariant ball in $\mathbb{R}^n$
- $\|x\|_p = \sqrt[p]{\sum_{i=1}^n |x_i|^p}$, $B_p^n = \{x : \|x\|_p \leq 1\}$
- μ_K — the uniform measure on K, $|\cdot|$ — the Lebesgue measure
- $x_1, x_2, \ldots, x_n$ denote the coordinates of a vector x
- $X = (X_1, X_2, \ldots, X_n)$ is the random variable equidistributed on K
- C, c are universal constants

An easy property

Let $(x_1, x_2, \ldots, x_n) \in K$, and $x_1 > x_2 > 0$. Then:

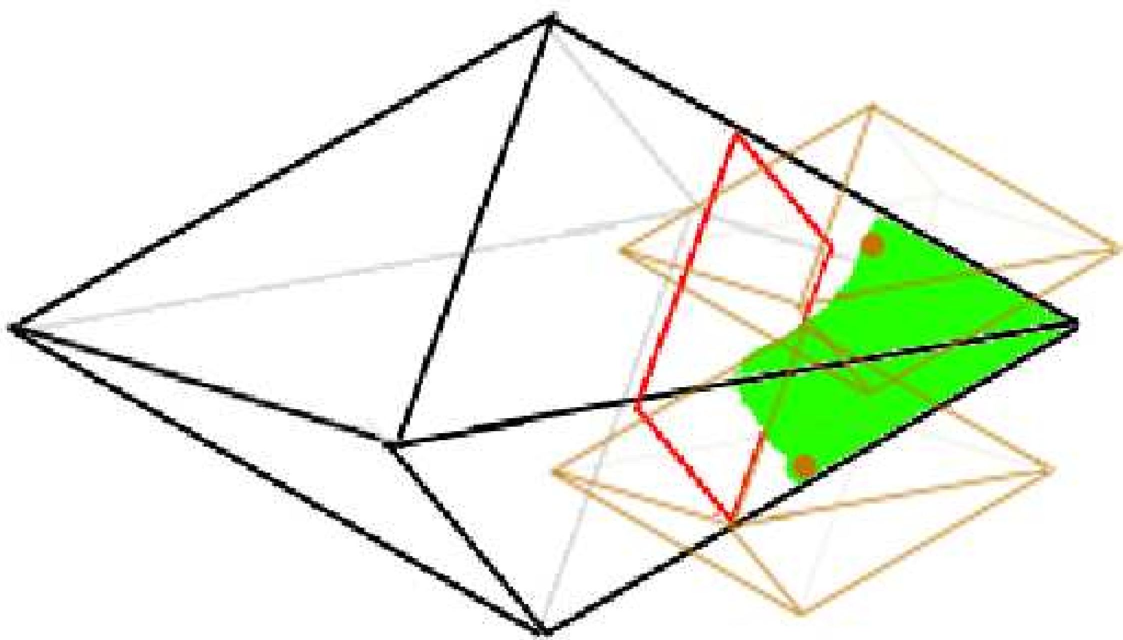
- $(x_2, x_1, x_3, \ldots, x_n) \in K$
- $(tx_1 + (1-t)x_2, tx_2 + (1-t)x_1, x_3, \ldots, x_n) \in K)$
- $(x_1 - a, x_2 + a, x_3, \ldots, x_n) \in K$ for $a > 0$ and $x_1 - a > x_2$
- $(x_1 - a, -(x_2 + a), x_3, \ldots, x_n) \in K$
- $(x_1 - a, x_2 + b, x_3, \ldots, x_n) \in K$ for $a > b > 0$

In particular, $K \subset \frac{n}{2} B_1^n$ (this is the correct order, $c(n) n B_1^n$ has volume 1 for $c(n) \simeq 1/2e$).

A concentration away from zero

We can prove:
Let $p \geq 1$, $t \geq 0$. Let $A \subset K$, suppose $\|x\|_p \geq 4pt\sqrt[p]{n}$ for $x \in A$, then

$$\mu_K(A + tB_1^n \qquad\qquad) \geq ce^{ct}\mu_K(A).$$



A concentration away from zero

We would like:
Let $p \geq 1$, $t \geq 0$. Let $A \subset K$,
then

$$\mu_K(A + tB_1^n + \sqrt{t}B_2^n) \geq ce^{ct}\mu_K(A).$$

The above is the “far away from zero” half of such a theorem.

Ingredients, part I (for B_1^n)

$$\mu_K(A + tB_1^n \geq ce^{ct}\mu_K(A).$$

Suppose $A \subset cnB_1^n$, $x_1 > t$ for $x \in A$. Let

$$S(x) = \Big(x_1 - t, (1+\frac{t}{cn})x_2, (1+\frac{t}{cn})x_3, \ldots, (1+\frac{t}{cn})x_n\Big).$$

Then:

- $S(A) \subset cnB_1^n$,
- $S(A) \subset A + 2tB_1^n$,
- $|S(A)| \simeq e^{ct}|A|$ ($\mathrm{d}etS = (1+t/cn)^{n-1}$).

Ingredients, part I (for K)

We will need $x_1 > t + 4$, and take $S(x) = (x_1 - t, f(x_2), f(x_3), \ldots, f(x_n))$ for

$$f(a) = \begin{cases} a + ta/n & |a| \in [0,2] \\ a + (4-a)t/n & |a| \in (2,4] \\ a & |a| > 4 \end{cases}$$

Then:

- $S(A) \subset K$,
- $S(A) \subset A + 2tB_1^n$,
- $|S(A)| \simeq e^{ct}|A|$ $(\det S > (1 - t/n)^{n/4}(1 + t/n)^{3n/4})$

Ingredients, part II

We will integrate $\|x\|_p^p d\mu_K$ by parts, using the previous statement:

$$\int_{A+tB_1^n} I_{\{|x_i|\geq s\}} d\mu_K(x) \geq e^{ct} \int_A I_{\{|x_i|\geq s+t\}} d\mu_K(x).$$

We receive a similar statement:

$$\int_{A+tB_1^n} \|x\|_p^p d\mu_K(x) \geq e^{ct} \int_A (\|x\|_p - t\sqrt[p]{n})_+^p d\mu_K(x).$$

$$\int \|x\|_p^p = \sum \int |x_i|^p = \sum \int p|t|^{p-1} I(|x_i| > t)$$

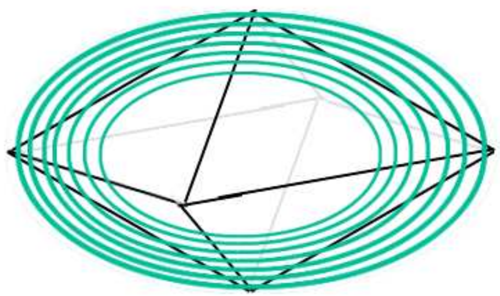
Ingredients, part II

We obtain (recall $\|x\|_p \geq Ctp\sqrt[p]{n}$):

$$\int_{A+tB_1^n} \|x\|_p^p d\mu_K(x) \geq e^{ct} \int_A (\|x\|_p - t\sqrt[p]{n})_+^p d\mu_K(x) \geq$$

$$e^{ct} \int_A \left(\frac{p-c}{p}\right)^p \|x\|_p^p d\mu_K(x)$$

Then split the set into parts where $\|x\|_p^p$ is $\pm$constant and deduce A enlarges:



Negative association (hypot.)

Definition 1. *K has neg. ass. of abs. values if for any coordinate–wise increasing functions f, g*

$$\mathbb{E}f(X_1, X_2, \ldots, X_k)g(X_{k+1}, \ldots, X_n) \leq \mathbb{E}f(X_1, X_2, \ldots, X_k)\mathbb{E}g(X_{k+1}, \ldots, X_n),$$

where $(X_1, X_2, \ldots, X_n)$ is uniformly distributed on K.

Holds for:

- K being an Orlicz ball
- A few other permutation invariant examples
- A few random computer–checked cases

Negative association (hypot.)

Definition 2. *K has neg. ass. of abs. values if for any coordinate–wise increasing functions f, g*

$$\mathbb{E}f(X_1, X_2, \ldots, X_k)g(X_{k+1}, \ldots, X_n) \le \mathbb{E}f(X_1, X_2, \ldots, X_k)\mathbb{E}g(X_{k+1}, \ldots, X_n),$$

where $(X_1, X_2, \ldots, X_n)$ is uniformly distributed on K.

Does not hold for:

- K being a general 1–symmetric body
- Permutation invariant log–concave measures

Easier versions (also open)

Definition 3. K *has negative ass. of abs. values if for any coordinate–wise increasing functions* f, g

$$\mathbb{E}f(X_1, X_2, \ldots, X_k)g(X_{k+1}, \ldots, X_n) \leq \mathbb{E}f(X_1, X_2, \ldots, X_k)\mathbb{E}g(X_{k+1}, \ldots, X_n).$$

Easier versions (also open)

Definition 4. K *has weak negative ass. of abs. values if for any coordinate–wise increasing functions* f, g

$$\mathbb{E}f(X_1)g(X_2, X_3, \ldots, X_n) \leq \mathbb{E}f(X_1)\mathbb{E}g(X_2, X_3, \ldots, X_n).$$

- One of the functions is of one variable only

Easier versions (also open)

Definition 5. K *has very weak negative ass. of abs. values if for any coordinate–wise increasing functions* f, g

$$\mathbb{E}f(X_1)g(X_2) \leq \mathbb{E}f(X_1)\mathbb{E}g(X_2).$$

- One of the functions is of one variable only
- Both of the functions are of one variable only

Easier versions (also open)

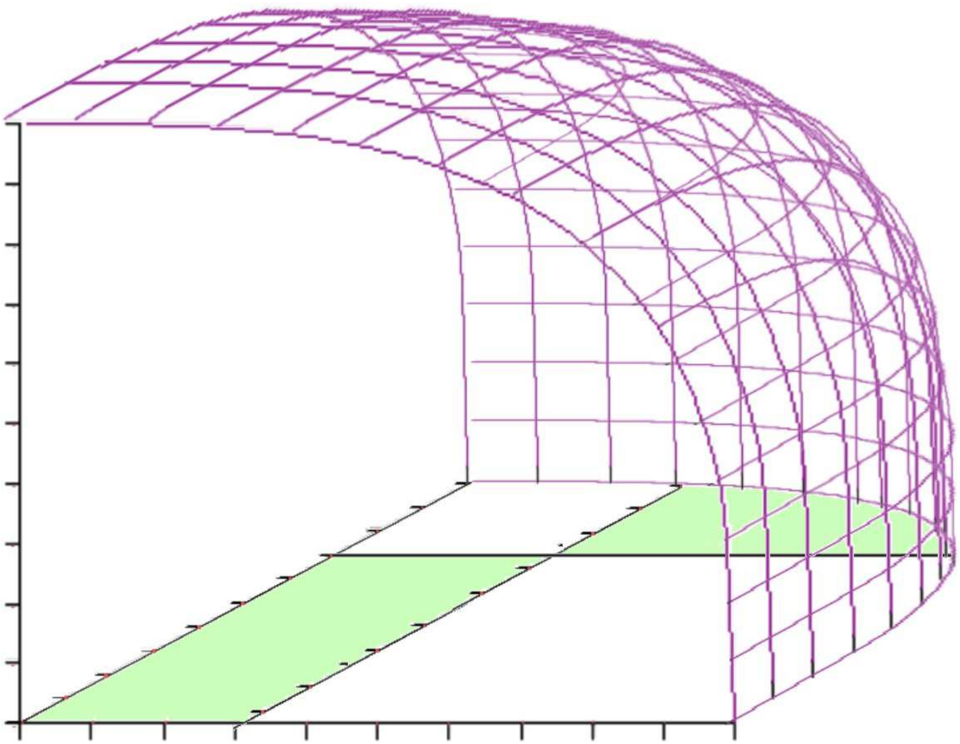
Definition 6. *K has very weak negative ass. of abs. values if for any coordinate–wise increasing functions f, g*

$$\mathbb{E}f(X_1)g(X_2) \leq \mathbb{E}f(X_1)\mathbb{E}g(X_2).$$

- One of the functions is of one variable only
- Both of the functions are of one variable only
- The three–dimensional case

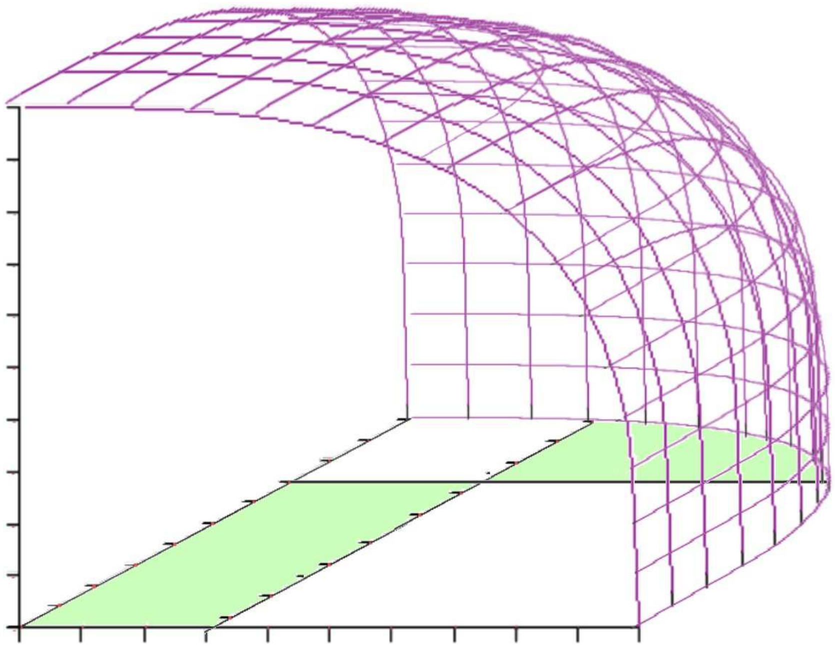
The three–dimensional case

$$\mathbb{P}(|X_1| > a, |X_2| > b) \leq \mathbb{P}(|X_1| > a)\mathbb{P}(|X_2| > b).$$



Very weak n.a.a.v for Orlicz balls

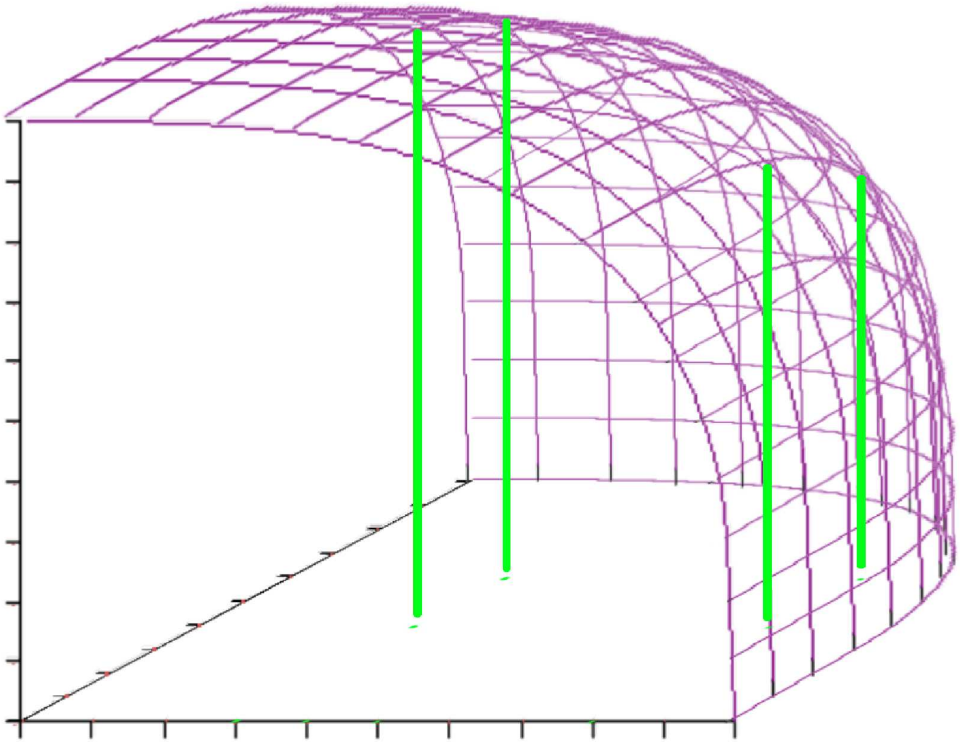
Restriction to probabilities:



$$\mathbb{P}(|X_1| > a, |X_2| > b)\mathbb{P}(|X_1| < a, |X_2| < b) \leq \mathbb{P}(|X_1| > a, |X_2| > b)\mathbb{P}(|X_1| < a, |X_2| > b).$$

Very weak n.a.a.v for Orlicz balls

Restriction to density of projection onto $\mathbb{R}^2$:

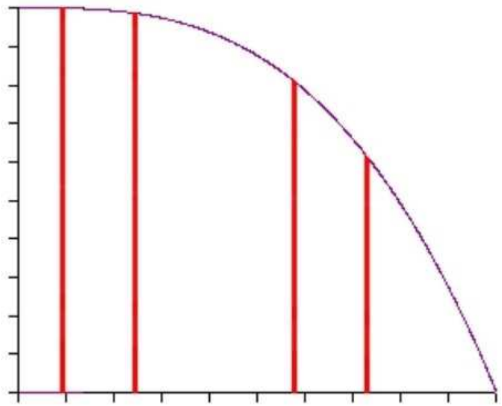


$$\{(z_3, \ldots, z_n) : \sum_{i=3}^{n} f_i(z_i) + f_1(x) + f_2(y) \leq n\}.$$

Very weak n.a.a.v for Orlicz balls

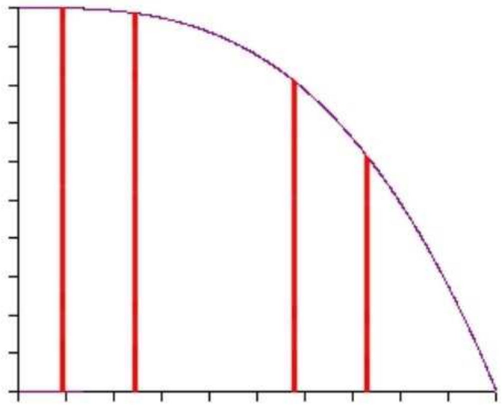
$$\{(z_3, \ldots, z_n) : \sum_{i=3}^{n} f_i(z_i) + f_1(x) + f_2(y) \leq n\}.$$

Investigating a different Orlicz ball (for functions $Id, f_3, f_4, \ldots, f_n$) at points $f_1(x_i) + f_2(y_j)$, $i, j \in \{1, 2\}$:



Very weak n.a.a.v for Orlicz balls

Apply the Brunn–Minkowski inequalty: $g(a)g(d) \leq g(b)g(c)$ if $a + d = b + c$ and $a < b < c < d$.



A side remark

Permutation invariant measures *are not* necessarily generated as limits of projections of permutation invariant bodies!
A recurring question is “starting with a certain class of measures, what can we get by applying tensor products, linear transformations and weak limits?”

- Ψ_2 behaviour
- Infimum convolution inequalities

No return to convexity

Theorem 7. *There is a class of measures containing all 1–dimensional log–concave measures, closed under linear maps, products and weak limits that contains no convex bodies except parallelotopes.*

There are probably many reasons for this phenomenon, tied basically to smoothness properties.

The end

Thank you for your attention.