On the isotropy constant of projections of polytopes

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(joint work with D. Alonso-Gutiérrez, J. Bastero and P. Wolff)

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Introduction

Let $K \subset \mathbb{R}^d$ be a symmetric convex body. Its isotropy constant L_K is

$$dL_K^2 := \text{mín}\left\{ \frac{1}{|TK|^{1+\frac{2}{d}}} \int_{TK} |x|^2 dx \;:\; T \in GL(d) \right\}$$

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- **Conjecture:** $L_K \leq L_{B_\infty^d} \simeq 0{,}29$

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For random convex bodies, (2008-2009)

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Theorem (Klartag-Kozma)

Let $G_0, \ldots, G_n$ be i.i.d. gaussian random vectors in $\mathbb{R}^d$, $n > d$. For $K = conv\{\pm G_0, \ldots, \pm G_n\}$ we have that with probability greater than $1 - Ce^{-cd}$

$$L_K \leq C.$$

(extra arguments needed for $n \leq cd$)

Back

Theorem (Alonso-Gutierrez and Dafnis, Giannopoulos, Guedon)

If $n \geq cd$ and $\{P_i\}_{i=0}^n$ are independent random points

- *on S^{d-1}*
- *on an isotropic unconditional convex body in $\mathbb{R}^d$*

and $K = conv\{\pm P_1, \ldots, \pm P_n\}$ then with probability $\geq 1 - c_1 e^{-c_2 d}$

$$L_K \leq C$$

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For any d and $1 < p \leq \infty$, (M. Junge, 1994, 1995; E. Milman, 2006)

- $L_{P_E(B_p^n)} \leq c\sqrt{p'}$, $\quad \frac{1}{p} + \frac{1}{p'} = 1$.
- $L_{P_E(B_1^n)} \leq c \log n$.

$P_E(B_1^n)$ is a d-dimensional convex polytope of at most n vertices.

An observation. Random projections of B_1^n

- Recall $G_{n,d} := \{E \subseteq \mathbb{R}^n : E \text{ subspace } d\text{-dimensional}\}$. There exists a unique probability measure, $\mu_{n,d}$ (Haar measure) on $G_{n,d}$ invariant under orthogonal transformations.

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- If G is an $n \times d$, $G : \mathbb{R}^d \to \mathbb{R}^n$, random matrix with entries g_{ij} i.i.d. standard gaussian r.v.'s then *by uniqueness* for every borelian $A \subset G_{n,d}$,

$$\mu_{n,d}(A) = \mathbb{P}\{\mathrm{Im}\ G \in A\}$$

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- If G is an $n \times d$ matrix and $E = \mathrm{Im}\ G \in G_{n,d}$ then (linear algebra)

$$G^t_{|E} P_E B_1^n = \mathrm{conv}\{\pm G_1, \ldots, \pm G_n\}$$

where G_i are the rows of G.

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$$\mu_{n,d}\{E \in G_{n,d} \ : \ L_{P_E B_1^n} \leq C\} > 1 - ce^{-c'd}$$

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- For d as small as $d < c \log n$ Dvoretzky's theorem says that $P_E B_1^n$ is typically like the euclidean ball B_2^d and so,

$$\mu_{n,d}\{E \in G_{n,d} \ : \ L_{P_E B_1^n} \leq C\} > 1 - ce^{-c' \log n}$$

An observation. Random projections of B_1^n

In conclusion, there exists $C, c, c' > 0$ such that for all $1 \leq d \leq n$,

$$\mu_{n,d}\{E \in G_{n,d} \ : \ L_{P_E B_1^n} \leq C\} > 1 - ce^{-c' \text{máx}\{d, \log n\}}$$

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On the other hand recall the following:

Fact

If $L_{P_E B_1^n} \leq C$ for every d-dimensional E and every $1 \leq d \leq n$, then $L_K \leq C$ for every symmetric convex body $K \subset \mathbb{R}^d$ in any dimension d.

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In this sense, "most" symmetric convex bodies $K \subseteq \mathbb{R}^d$ have isotropy constant bounded.

Main result. Projections of B_1^n

Theorem

$$L_{P_E(B_1^n)}, L_{P_E(S_n)} \leq C\sqrt{\frac{n}{d}}$$

where S_n is the n-dimensional regular simplex.

Main result. Projections of B_1^n

Theorem

$$L_{P_E(B_1^n)}, L_{P_E(S_n)} \leq C\sqrt{\frac{n}{d}}$$

where S_n is the n-dimensional regular simplex.

$$\Updownarrow$$

Corollary

Let $K \subseteq \mathbb{R}^d$ be a convex polytope with n vertices. Then

$$L_K \leq C\sqrt{\frac{n}{d}}$$

Proof: Starting point

For any convex polytope $K \subset \mathbb{R}^n$,

$$dL^2_{P_E(K)} \leq \frac{1}{|P_E(K)|^{2/d}} \frac{1}{|P_E(K)|} \int_{P_E(K)} |x|^2 \, \lambda_E(dx)$$

Proof: Starting point

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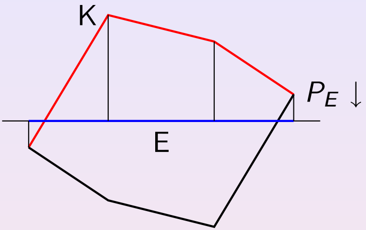
$$dL^2_{P_E(K)} \leq \frac{1}{|P_E(K)|^{2/d}} \frac{1}{|P_E(K)|} \int_{P_E(K)} |x|^2 \, \lambda_E(dx)$$

- Estimate from above $\frac{1}{|P_E(K)|} \int_{P_E(K)} |x|^2 \, \lambda_E(dx)$
- Estimate from below $|P_E(K)|$

Integrating on $P_E(K)$



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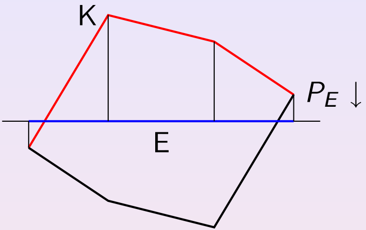
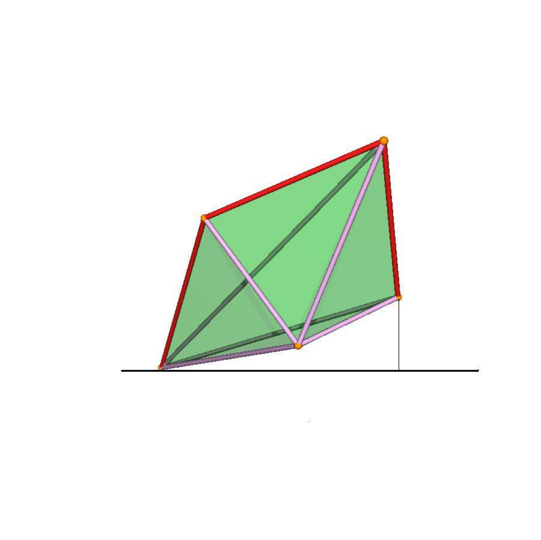


Main Lemma

Let $K \subset \mathbb{R}^n$ be a convex polytope and E a d dimensional subspace of $\mathbb{R}^n$. There exists a subset $\tilde{\mathcal{F}}$ of d dimensional faces of K such that

- $\{P_E(\operatorname{relint} F) \mid F \in \tilde{\mathcal{F}}\}$ is a family of pair-wise disjoint sets,
- $\bigcup\{P_E(F); F \in \tilde{\mathcal{F}}\} = P_E(K), \quad a.e.\ \lambda_E$
- For each $F \in \tilde{\mathcal{F}}$, $P_E|F \colon F \to P_E(F)$ is an affine isomorphism.

Integrating on $P_E(K)$



Integrating on $P_E(K)$

Corollary

Let $K \subset \mathbb{R}^n$ be a convex polytope and E a d dimensional subspace of $\mathbb{R}^n$. There exists a subset $\tilde{\mathcal{F}}$ of d dimensional faces of K such that for any integrable function $f\colon P_E(K) \to \mathbb{R}$,

$$\int_{P_E(K)} f(x)\,\lambda_E(dx) = \sum_{F \in \tilde{\mathcal{F}}} \int_{P_E(F)} f(x)\,\lambda_E(dx) = $$
$$= \sum_{F \in \tilde{\mathcal{F}}} \frac{|P_E(F)|}{|F|} \int_F f(P_E y)\,\lambda_{\mathrm{aff} F}(dy)$$

In particular for $f \equiv 1$, $\quad |P_E(K)| = \sum_{F \in \tilde{\mathcal{F}}} |P_E(F)|$

Sketch of the proof

$$\frac{1}{|P_E(K)|}\int_{P_E(K)}|x|^2\,dx=\frac{1}{|P_E(K)|}\sum_{F\in\tilde{\mathcal{F}}}\int_{P_E(F)}|x|^2\,dx=$$

Sketch of the proof

$$\frac{1}{|P_E(K)|}\int_{P_E(K)} |x|^2\,dx = \frac{1}{|P_E(K)|}\sum_{F\in\tilde{\mathcal{F}}}\int_{P_E(F)} |x|^2\,dx =$$

$$\frac{1}{|P_E(K)|}\sum_{F\in\tilde{\mathcal{F}}}\frac{|P_E(F)|}{|F|}\int_F |P_E y|^2\,dy \le \sum_{F\in\tilde{\mathcal{F}}}\frac{|P_E(F)|}{|P_E(K)|}\frac{1}{|F|}\int_F |y|^2\,dy$$

Sketch of the proof

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$$\frac{1}{|P_E(K)|}\sum_{F\in\tilde{\mathcal{F}}}\frac{|P_E(F)|}{|F|}\int_F |P_E y|^2\,dy \le \sum_{F\in\tilde{\mathcal{F}}}\frac{|P_E(F)|}{|P_E(K)|}\frac{1}{|F|}\int_F |y|^2\,dy$$

$$\le \sup_{F\in\tilde{\mathcal{F}}}\frac{1}{|F|}\int_F |y|^2\,dy \qquad \left(\text{since} \quad 1 = \sum_{F\in\tilde{\mathcal{F}}}\frac{|P_E(F)|}{|P_E(K)|}\right)$$

In the symmetric case, $K = B_1^n$, denoting Δ_d the d-dimensional regular simplex in $\langle e_1, \ldots, e_{d+1} \rangle$,

$$\frac{1}{|P_E(B_1^n)|} \int_{P_E(B_1^n)} |x|^2 \, dx \leq \sup_{F \in \tilde{\mathcal{F}}} \frac{1}{|F|} \int_F |y|^2 \, dy = \frac{1}{|\Delta_d|} \int_{\Delta_d} |x|^2 \, dx = \frac{2}{d+2}$$

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On the other hand, $n^{-1/2} B_2^n \subseteq B_1^n \Longrightarrow n^{-1/2} P_E(B_2^n) \subseteq P_E(B_1^n)$. Therefore

$$|P_E(B_1^n)|^{1/d} \geq \frac{c}{\sqrt{nd}}$$

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$$|P_E(B_1^n)|^{1/d} \geq \frac{c}{\sqrt{nd}}$$

Combining all estimates we get

$$L^2_{P_E(B_1^n)} \leq \frac{1}{d} \frac{1}{|P_E(B_1^n)|^{2/d}} \frac{1}{|P_E(B_1^n)|} \int_{P_E(B_1^n)} |x|^2 \, dx \leq \frac{1}{d} \frac{nd}{c^2} \frac{2}{d+2} \leq c' \frac{n}{d}$$

Application

Our result,

Theorem

Let $K \subseteq \mathbb{R}^d$ be a convex polytope with n vertices. Then

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Theorem

Let $K \subseteq \mathbb{R}^d$ be a convex polytope with n vertices. Then

$$L_K \leq C\sqrt{\frac{n}{d}}$$

solves the difficulties in the papers above ▸ Rdm:

If the number of vertices is proportional to the dimension, $n \leq cd$ then $L_K \leq C$ *deterministically*.

The isotropy constant of projections of $B_p^n, 1 < p \leq 2$.

For hyperplanes,

Theorem.

There exists $C > 0$ such that for every $1 < p \leq 2$ and any hyperplane $H = \theta^{\perp}$,

$$L_{P_H(B_p^n)} \leq C$$

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Proof of the Theorem. (Ideas from Barthe-Naor). From the definition,

$$(n-1)L^2_{P_H(B_p^n)} \leq \frac{1}{|P_H(B_p^n)|^{\frac{2}{n-1}}} \cdot \frac{1}{|P_H(B_p^n)|} \int_{P_H(B_p^n)} |x|^2 \, dx$$

The isotropy constant of hyperplane projections of $B_p^n, 1 < p \leq 2$.

The first step is **Cauchy's formula:** For good f we have

$$\int_{P_H(B_p^n)} f(x)dx = \frac{1}{2}|\partial B_p^n| \int_{\partial B_p^n} f(P_H(y))|\langle N(y), \theta\rangle| d\sigma_p^n(y)$$

where $N(y)$ is the unit normal vector to ∂B_p^n and σ_p^n the normalized area measure. We apply it to $f = 1$ and $f(y) = |y|^2$.

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Question. In order to extend the proof to lower dimensional projections, E, we need a formula for

$$\int_{P_E(B_p^n)} f(x)dx$$

The proof

Fact 1 Naor, Romik 2003. The relation between two measures on ∂B_p^n: σ_p^n (the normalized area measure) and μ_p^n (the cone probability measure) is given by

$$\frac{d\sigma_p^n}{d\mu_p^n}(x) = \frac{n|B_p^n|}{|\partial B_p^n|} \left|\bigtriangledown(\|\cdot\|_p)(x)\right| \qquad a.e. \;\; x \in \partial B_p^n$$

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Fact 1 Naor, Romik 2003. The relation between two measures on ∂B_p^n: σ_p^n (the normalized area measure) and μ_p^n (the cone probability measure) is given by

$$\frac{d\sigma_p^n}{d\mu_p^n}(x) = \frac{n|B_p^n|}{|\partial B_p^n|} |\nabla(\|\cdot\|_p)(x)| \qquad a.e. \;\; x \in \partial B_p^n$$

Fact 2 Schechtmann, Zinn 1990. There is a representation of the cone measure μ_p^n. Let $g_1, \dots, g_n$ be i.i.d. copies of a r.v. with density $c_p e^{-|t|^p}$, $t \in \mathbb{R}$. Define $S = (\sum_{i=1}^n |g_i|^p)^{1/p}$. *Then the probability determined by* $(g_1/S, \dots, g_n/S) \in \partial B_p^n$ *is* μ_p^n. *Moreover,* $(g_1/S, \dots, g_n/S)$ **is independent of** S**.**

By Cauchy's formula and good f,

$$\int_{P_H(B_p^n)} f(x)dx = \frac{1}{2}|\partial B_p^n| \int_{\partial B_p^n} f(P_H(y))|\langle N(y), \theta\rangle| d\sigma_p^n(y)$$

$$(N(y) \text{ is the unit normal vector to } \partial B_p^n)$$

$$= \frac{n}{2}|B_p^n| \int_{\partial B_p^n} f(P_H(y))|\langle \bigtriangledown(\|\cdot\|_p)(y), \theta\rangle| d\mu_p^n(y)$$

$$= \frac{n}{2}|B_p^n| \int_{\partial B_p^n} f(P_H(y)) \left| \sum_{i=1}^n |y_i|^{p-1} \mathrm{sgn}(y_i)\theta_i \right| d\mu_p^n(y)$$

Use it with $f = 1$ (for the volume) and $f(y) = |y|^2$ (and also $|P_H(y)| \leq |y|$).

Now use the representation of $d\mu_p^n$. $y_i \to \frac{g_i}{S}$.

$$\frac{1}{|P_H(B_p^n)|}\int_{P_H(B_p^n)}|x|^2dx \le \frac{\mathbb{E}\sum_{i=1}^n \frac{|g_i|^2}{S^2}\left|\sum_{i=1}^n \frac{|g_i|^{p-1}}{S^{p-1}}\mathrm{sgn}(g_i)\theta_i\right|}{\mathbb{E}\left|\sum_{i=1}^n \frac{|g_i|^{p-1}}{S^{p-1}}\mathrm{sgn}(g_i)\theta_i\right|}$$

$$= \text{ (by independence)}$$

$$= \frac{\mathbb{E}S^{p-1}}{\mathbb{E}S^{p+1}}\sum_{i=1}^n \frac{\mathbb{E}|g_i|^2\left|\sum_{i=1}^n |g_i|^{p-1}\mathrm{sgn}(g_i)\theta_i\right|}{\mathbb{E}\left|\sum_{i=1}^n |g_i|^{p-1}\mathrm{sgn}(g_i)\theta_i\right|}$$

Since $S = (\sum_{i=1}^n |g_i|^p)^{1/p}$,

$$\frac{\mathbb{E}S^{p-1}}{\mathbb{E}S^{p+1}} \le \frac{c}{n^{2/p}}$$

$$
\begin{aligned}
\mathbb{E}\left|\sum_{i=1}^{n}|g_i|^{p-1}\mathrm{sgn}(g_i)\theta_i\right| &= \mathbb{E}_\varepsilon\mathbb{E}_g\left|\sum_{i=1}^{n}|\varepsilon_i g_i|^{p-1}\varepsilon_i\mathrm{sgn}(g_i)\theta_i\right| \\
&\geq \text{ (by Khinchine's inequality)} \\
&\geq C\ \mathbb{E}_g\left(\sum_{i=1}^{n}|g_i|^{2p-2}\theta_i^2\right)^{1/2} \\
&\geq \text{ (by Jensen's inequality)} \\
&\geq C\ \mathbb{E}\sum_{i=1}^{n}|g_i|^{p-1}\theta_i^2 = C\ \mathbb{E}|g|^{p-1} \geq C > 0
\end{aligned}
$$

With the same argument,

$$
\mathbb{E}|g_i|^2\left|\sum_{i=1}^{n}|g_i|^{p-1}\mathrm{sgn}(g_i)\theta_i\right| \leq C
$$

The isotropy constant of hyperplane projections of $B_p^n, 1 < p \leq 2$.

Putting all estimates together, $\dfrac{1}{|P_H(B_p^n)|}\displaystyle\int_{P_H(B_p^n)} |x|^2 dx \leq C\frac{n}{n^{2/p}}$

By the same argument, $|P_H(B_p^n)|^{\frac{2}{n-1}} \geq \frac{C}{n^{2/p}}$ and so,

$$(n-1)L^2_{P_H(B_p^n)} \leq \frac{1}{|P_H(B_p^n)|^{\frac{2}{n-1}}} \cdot \frac{1}{|P_H(B_p^n)|}\int_{P_H(B_p^n)} |x|^2\, dx \leq Cn$$

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