On the volume of random polytopes generated by points in an isotropic convex body

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- K - a convex body in $\mathbb{R}^n$
- $|\cdot|, \langle \cdot, \cdot \rangle$ - canonical Euclidean norm and inner-product.
- n is large
- c, c_1, C, C_1, C', etc - absolute constants (independent of K and n).

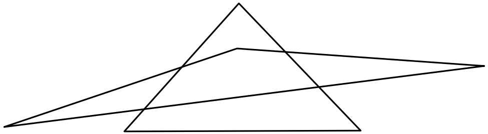
- We assume $\mathrm{vol}\,(K) = 1$ and treat K as probability space.
- X - random vector distributed uniformly in K, i.e.,

$$\mathbb{P}\,(X \in A) = \mathrm{vol}\,(K \cap A)\,.$$

- Suppose $\mathrm{vol}(K) = 1$ and the center of mass of K is 0.

Problem: Find $T \in SL(n)$ such that

$$\int_{TK} |x|^2 dx \leq \int_{SK} |x|^2 dx \qquad \forall\, S \in SL(n). \tag{1}$$



Fact: Let $T \in SL(n)$. Then (1) holds if and only if there is a constant L_K such that

$$\int_{TK} \langle x, \theta \rangle^2 dx = L_K^2$$

for any $\theta \in S^{n-1}$.

A convex body $K \subset \mathbb{R}^n$ is *isotropic* if $\mathrm{vol}(K) = 1$, the center of mass of K is 0 and there is a constant L_K such that

$$\int_K \langle x, \theta \rangle^2 dx = L_K^2 \quad \forall\, \theta \in S^{n-1}.$$

- L_K is called the *isotropic constant*.
- Any convex body has an affine image which is isotropic.

If K is isotropic, then for any subspace $E \subset \mathbb{R}^n$,

$$\int_K |P_E x|^2 dx = \dim(E) L_K^2.$$

In particular,

$$\left(\int_K |x|^2 dx \right)^{1/2} = \sqrt{n} L_K.$$

Survey on isotropic convex bodies: [Milman-Pajor, 1989].

Estimating L_K

Fact: For any convex body $K \subset \mathbb{R}^n$,

$$L_K \geq L_{B_2^n} = \frac{1}{\sqrt{2\pi e}}.$$

Conjecture:

$$\sup\{L_K : K \subset \mathbb{R}^n, n \in \mathbb{N}\} \leq C.$$

Equivalent Formulation: Slicing Problem: If $\mathrm{vol}\,(K) = 1$, is there a hyperplane H such that

$$\mathrm{vol}\,(K \cap H) \geq c?$$

Best Known Bound and Positive Results

Results for general convex body $K \subset \mathbb{R}^n$

[Bourgain, 1990]: $L_K \leq C_1 n^{1/4} \log n$.

[Klartag, 2006]: $L_K \leq C_2 n^{1/4}$.

Positive answers for many classes of convex bodies, e.g.,:

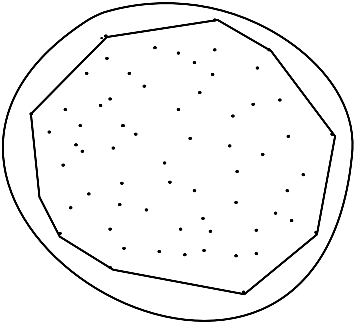
- Unconditional convex bodies [Bourgain, 1986], [Milman-Pajor, 1988]
- Zonoids and duals of zonoids [Ball, 1991]
- Unit balls of Schatten norms [König-Meyer-Pajor, 1998]
- ψ_2-bodies [Bourgain, 2002]
- Various random polytopes [Klartag-Kozma, 2007], [Dafnis-Giannopoulos-Guedon, 2008]

Random Polytopes

- K - isotropic convex body in $\mathbb{R}^n$
- $X_1, \dots, X_N$ - independent random vectors in K

Set

$$K_N := \mathrm{conv}\left\{X_1, \dots, X_N\right\}.$$



L_K and the volume of random polytopes

Theorem (Dafnis-Giannopoulos-Tsolomitis, 2009)

For all $n+1 \leq N \leq e^n$,

$$c_1 \frac{\sqrt{\log \frac{2N}{n}}}{\sqrt{n}} \leq \mathrm{vol}\,(K_N)^{1/n} \leq C_1 \frac{\sqrt{\log \frac{2N}{n}}}{\sqrt{n}} L_K$$

with probability at least $1 - 1/N$.

Conjecture: For all $n+1 \leq N \leq e^n$,

$$c_1' \frac{\sqrt{\log \frac{2N}{n}}}{\sqrt{n}} L_K \leq \mathbb{E}\,\mathrm{vol}\,(K_N)^{1/n}.$$

$N = n + 1$: treated in [Dafnis, Giannopoulos, Guedon, 2008]:

$$\mathrm{vol}\,(K_{n+1}) \geq \left(\frac{c}{n}\right)^{n/2}$$

with probability at least $1 - e^{-n/\log n}$.

Proposition (P. 2009)

Let $K^{(1)}, \ldots, K^{(n+1)}$ be isotropic convex bodies in $\mathbb{R}^n$. Let $X_1, \ldots, X_{n+1}$ be independent random vectors such that X_i is uniformly distributed in $K^{(i)}$ for $i = 1, \ldots, n+1$. Then

$$\mathrm{vol}\,(\mathrm{conv}\,\{X_1, \ldots, X_{n+1}\}) \geq \left(\frac{c_1}{n}\right)^{n/2} \prod_{i=1}^{n+1} L_{K^{(i)}},$$

with probability at least $1 - e^{-n}$.

Symmetric case:

$$\mathrm{vol}\left(\mathrm{conv}\left\{\pm X_1, \dots, \pm X_n\right\}\right) = \frac{2^n}{n!}|\det[X_1 \cdots X_n]|$$

$$|\det[X_1 \cdots X_n]| = |X_1||P_{F_1^\perp} X_2| \cdots |P_{F_{n-1}^\perp} X_n|,$$

where

$$F_0 = \{0\}, \qquad F_k := \mathrm{span}\{X_1, \dots, X_k\}$$

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Lemma (follows from (Guedon, 1999))

Let X be a random vector distributed uniformly in an isotropic convex body $K \subset \mathbb{R}^n$. Let $1 \le \ell \le n$ and let $E \subset \mathbb{R}^n$ be a subspace with $\dim E = \ell$. *Then*

$$Y := \frac{|P_E X|}{\sqrt{\ell} L_K}$$

satisfies

$$\mathbb{E}|Y|^{-1/2} \le C.$$

The case of the ball

The same argument for B_2^n gives an immediate proof of a known formula:

Proposition

Let $X_1, \dots, X_n$ be independent random vectors uniformly distributed in the Euclidean ball of volume one $\overline{B_2^n}$. Then for any $q \in (-1, \infty)$,

$$\mathbb{E}|\mathrm{conv}\,\{\pm X_i\}_{i=1}^n|^q = \left(\frac{2^n \Gamma(1+\frac{n}{2})}{\pi^{n/2} n!}\right)^q \left(\frac{\Gamma(1+\frac{n}{2})}{\Gamma(1+\frac{n+q}{2})}\right)^n \prod_{k=1}^n \frac{\Gamma(\frac{k+q}{2})}{\Gamma(\frac{k}{2})}.$$

Similar formulas appear in

[Miles, 1971], [Ruben, 1979], [Mathai, 1999], [Meckes, 2004]...

Zonotopes

A *zonotope* is a (Minkowski) sum of line segments:

$$\sum_{i=1}^{N} [0, x_i],$$

where the $x_i \in \mathbb{R}^n$ and

$$[0, x_i] = \{tx_i : 0 \leq t \leq 1\}.$$

Fact:

$$\mathrm{vol}\left(\sum_{i=1}^{N} [0, x_i]\right) = \sum_{\substack{I \subset \{1,\ldots,N\} \\ |I|=n}} \left|\det[x_i]_{i \in I}\right|.$$

$V_1, V_2, \ldots, V_N \subset \mathbb{R}^n$ - convex bodies of volume 1, $p > 0$.

$$\mathcal{I}_p(V_1, \ldots, V_N) := \int_{V_1} \cdots \int_{V_N} \mathrm{vol}\left(\sum_{i=1}^{N} [0, x_i] \right)^p dx_1 \cdots dx_N$$

Theorem (Bourgain-Meyer-Milman-Pajor, 1988)

- $\mathcal{I}_p(V_1, \ldots, V_N) \geq \mathcal{I}_p(\overline{B_2^n}, \ldots, \overline{B_2^n})$
- $\mathcal{I}_{1/n}(\overline{B_2^n}, \ldots, \overline{B_2^n}) \geq \frac{c_1 N}{\sqrt{n}}$

Proposition (P. 2009)

Let $V_1, \ldots, V_N$ be isotropic convex bodies in $\mathbb{R}^n$. Then

$$\mathcal{I}_{1/n}(V_1, \ldots, V_N) \geq \frac{c_1 N}{\sqrt{n}} \left(\frac{1}{\binom{N}{n}} \sum_{\substack{I \subset \{1, \ldots, N\} \\ |I| = n}} \left(\prod_{i \in I} L_{V_i} \right)^{1/n} \right).$$

Assume $K = -K$, $V_i = -V_i$, $i = 1, \dots, N$. For $\lambda \in \mathbb{R}^N$, let

$$\|\lambda\| := \int_{V_1} \cdots \int_{V_N} \left\| \sum_{i=1}^{N} \lambda_i x_i \right\|_K dx_N \dots dx_1.$$

Theorem (Bourgain-Meyer-Milman-Pajor, 1988)

If $N = n$, $V_i = V$, $|V_i| = |K| = 1$ and $(\mathbb{R}^n, \|\cdot\|_K)$ has cotype, then

$$\|\lambda\| \geq \overline{c}\sqrt{n} \left(\prod_{i=1}^{n} |\lambda_i| \right)^{1/n} L_V.$$

Recent developments: [Gluskin-Milman, 2004]

Proposition (P. 2009)

Let $V_1, \dots, V_N$ be isotropic, $|K| = 1$. Then, for $\lambda \in \mathbb{R}^N$,

$$\|\lambda\| \geq c\sqrt{N} \left(\binom{N}{n}^{-1} \sum_{|I|=n} \left(\prod_{i \in I} |\lambda_i| \right)^{1/n} \left(\prod_{i \in I} L_{V_i} \right)^{1/n} \right).$$

Hadamard's inequality for random matrices

Let $A = [A_1 \cdots A_n]$. Hadamard's inequality:

$$|\det A| \leq \prod_{i=1}^{n} |A_i|$$

with equality iff the A_i are orthogonal. Set

$$h(A) := \frac{|\det A|}{\prod_{i=1}^{n} |A_i|}$$

Theorem (Dixon, 1984)

Let A_i be i.i.d. random vectors on S^{n-1}. Then for each $\varepsilon > 0$,

$$\lim_{n\to\infty} \mathbb{P}\left(n^{-1/4-\varepsilon} e^{-n/2} \leq h(A) \leq n^{-1/4+\varepsilon} e^{-n/2}\right) = 1.$$

Proposition (P. 2009)

Let $V_1, \dots, V_n$ be isotropic convex bodies in $\mathbb{R}^n$. Suppose that $A_1, \dots, A_n$ are independent random vectors such that A_i is distributed uniformly in V_i for $i = 1, \dots, n$. Let A be the matrix $A = [A_1 \cdots A_n]$. Then

$$\mathbb{P}\left(h(A)^{1/n} \in [c_1', c_2']\right) \geq 1 - 2e^{-c_3' n}$$

where $0 < c_1' < c_2' < 1$ and $c_3' > 0$ are absolute constants.

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