Some applications of moments in RMT

① Preliminaries

Let μ be a measure on $\mathbb{R}$; denote

$$m_k(\mu) = \int x^k \, d\mu \, , \quad k = 0, 1, 2, \ldots$$

We assume that these numbers (the <u>moments</u> of μ) are finite.

Problem of moments:

given $m_0, m_1, \ldots, m_{\bar{n}}$ ($\bar{n}$ may be finite or $+\infty$),

(a) does there exist μ so that $\int x^k d\mu = m_k$?

(b) what can one say about μ, e.g.

$$\max_\mu \mu(-\infty, x], \quad \min_\mu \mu(-\infty, x].$$

(a): "easy": $\Leftrightarrow$ for every positive polynomial $P(x) = \sum_{k=0}^{n} a_k x^k$, $\sum a_k m_k \geq 0$ ($\Leftrightarrow$ some determinants are positive).

(b): harder and more useful. Typically, we want to extract information about μ from $m_0, \ldots, m_n$ (which are precisely or approximately known). [Origin: Chebyshev's approach to the CLT, Markov, ... Stieltjes, ...]

Prop. 1 If μ is supported on a finite interval and $m_k(\nu) = m_k(\mu)$, $k = 0, 1, 2, \ldots$, then $\nu = \mu$.

Proof (a) $m_k(\mu) \leq C^k m_0(\mu)$, hence ν is also supported on $[-C, C]$.

(b) Any continuous fn f on $[-C, C]$ can be approximated by pol-s (in uniform topology). Hence

$$\int f d\nu = \int f d\mu, \quad f \in C[-C, C] \Rightarrow \nu = \mu.$$

$\blacksquare$

Prop. 2 If μ is such that $\int e^{\varepsilon|x|}\,d\mu(x) < \infty$ for some $\varepsilon > 0$, and $m_k(\nu) = m_k(\mu)$, $k = 0, 1, 2, \dots$, then $\nu = \mu$.

Hint if $\int e^{i\xi x}\,d\mu(x) = \int e^{i\xi x}\,d\nu(x)$, $\xi \in \mathbb{R}$, then $\mu = \nu$.

More precise criteria are available, esp.: Carleman's condition in the positive, and Krein's condition in the negative. See e.g. Akhiezer's book.

Let μ be a measure with finite moments. Orthogonalise $1, x, x^2, x^3, \dots$ in $L^2(\mu)$. We obtain the sequence $P_0(x), P_1(x), \dots$ of orthogonal pol-s (w.r. to μ).

Ex. ① $\mu = \gamma'$ (the standard Gaussian measure)
$\rightsquigarrow$ Hermite pol-s

② Wigner's measure $d\mu_W(x) = \frac{2}{\pi}(1-x^2)_+^{1/2}\,dx$

$\rightsquigarrow$ Chebyshev pol-s of the second kind,
$U_k(\cos\theta) = \sin((k+1)\theta)/\sin\theta$

hint Check that $\int U_k U_\ell \, d\mu_W = \delta_{k\ell}$ using the change of variables $x = \cos\theta$.

③ The Kesten-McKay measure $d\mu^d_{KM}(x) = \frac{2d(d-1)}{\pi}\,\frac{(1-x^2)_+^{1/2}\,dx}{d^2 - 4(d-1)x^2}$
$\rightsquigarrow$ $P_k(x) = \sqrt{\frac{d-1}{d}}\left[U_k(x) - \frac{1}{d-1}U_{k-2}(x)\right]$.

Prop. 3 $x P_k(x) = A_k P_{k+1}(x) + B_k P_k(x) + C_k P_{k-1}(x)$

Hint: by induction

Rmk For μ_W (and hence also for μ_{KM}), $A_k = \frac{1}{2}$, $B_k = 0$, $C_k = \frac{1}{2}$.

<u>Digression</u> A_k, B_k, C_k can be written in a matrix (Jacobi matrix):

$$J = \begin{pmatrix} B_1 & C_2 & & \\ A_1 & B_2 & C_3 & \\ & A_2 & B_3 & \\ & & & \ddots \end{pmatrix}$$

J represents multiplication by x in $L^2(\mu)$.

(Direct) spectral problem: given A_k, B_k, C_k, find μ

(inverse) spectral problem: given μ, find A_k, B_k, C_k

Sometimes (in RMT), one wishes to find the spectral measure of an operator H, and one manages to find a tridiagonal matrix isospectral to it. Then one needs to solve the direct spectral problem.

See e.g. Dumitriu & Edelman, and further works.

② Random matrices: the global regime

Given an $N \times N$ matrix H, set $\mu_H = \frac{1}{N} \sum \delta_{\lambda_i}$, where λ_i are the eigenvalues of H. If H is Hermitian, this is a measure on $\mathbb{R}$.

Now let H_N be a sequence of matrices of increasing size. Set $\mu_N = \mu_{H_N}$. Question: $\mu_N \xrightarrow[N\to\infty]{w} ?$

Next, assume that H_N are random. Then μ_N are also random (ie prob. distributions on the space of measures).

<u>Q-n</u> $\mu_N \xrightarrow{w,D} ?$ Is the limit random or deterministic?

<u>A</u> Wigner matrices

- $\{H_N(u,v) \mid u \le v\}$ are independent
- $\mathbb{E} H_N(u,v) = 0$; if $u<v$, $\mathbb{E} H_N(u,v)^2 = 1$
- $\mathbb{E} |H_N(u,v)|^k \le A_k < \infty$.

We shall consider the spectral measures μ_N of $\frac{H_N}{2\sqrt{N}}$ (this is just scaling).

<u>Thm</u> (Wigner) $\mu_N \xrightarrow{w,D} \mu_W$ (the limit is not random).

Plan of proof:

(i) Consider $\mathbb{E}\mu_N$ and prove that $\int x^k \, d\mathbb{E}\mu_N \to \int x^k \, d\mu_W$ for $k = 0, 1, 2, \ldots$

In particular, the prob. measures $\mathbb{E}\mu_N$ have unif. bdd. exp.-n and variance $\Rightarrow$ $\{\mathbb{E}\mu_N\}$ is precompact. By Prop. 1, every part. limit is equal to μ_W $\Rightarrow$ $\mathbb{E}\mu_N \xrightarrow{w} \mu_W$.

(ii) $\operatorname{Var} \int x^k d\mu_N \to 0$, $k = 0, 1, 2, \dots$ (and [illegible] $\mathbb{E} \int x^k d\mu_N \int x^\ell d\mu_N \to m_k(\mu_W) \cdot m_\ell(\mu_W)$)

therefore μ_N are concentrated around the mean $\mathbb{E}\mu_N$, and hence also $\mu_N \xrightarrow[N\to\infty]{w,P} \mu_W$.

Now to business.

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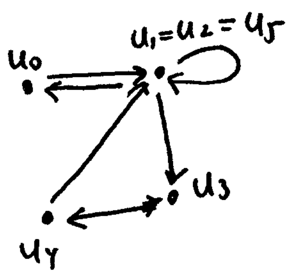
<u>Proof</u> $\int x^k \, d\,\mathbb{E}\mu_N = \mathbb{E} \int x^k d\mu_N = \mathbb{E} \frac{1}{N} \operatorname{tr}\left(\frac{H_N}{2\sqrt{N}}\right)^k =$

$$= \frac{1}{2^k} \cdot \frac{1}{N^{1+k/2}} \sum \mathbb{E} H_N(u_0, u_1) H_N(u_1, u_2) \cdots H_N(u_{k-1}, u_0).$$

Every addend corresponds to a path p_k in the complete graph K_N:



$= p_k$ (e.g. 7 4 4 2 8 4 7

We group the paths into equiv. classes with respect to the action of S_N.

<u>NB</u> The number of equivalence classes depends only on k (and not on N).

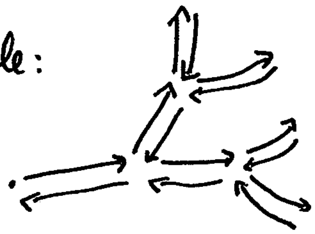
On every edge $e = (u, v)$, we have a r.v. $H_N(e) = H_N(u, v)$ $(= H_N(v, u))$. If and edge e appears $k(e)$ times,

$$\mathbb{E} H_N(u_0, u_1) \cdots H_N(u_{k-1}, u_0) = \prod_e \mathbb{E} H_N(e)^{k(e)}.$$

In particular, if $k(e) = 1$, the expectation is zero.

Consider the graph $G = (V, E)$ induced by p_k; it is connected, hence $\#V \leq \#E + 1 \leq \frac{k}{2} + 1$, with equality for trees

Example:



Thus the contr. of the equiv. class of G is

$$\leq N^{V} \cdot \frac{1}{2^k N^{1+k/2}} \cdot A_k \leq A_k' ;$$

If G is not a tree: $\leq \frac{A_k'}{N} \xrightarrow[N\to\infty]{} 0$. Thus we only consider trees.

<u>Fact</u> # tree-like paths of length k $= \begin{cases} 0, & k \text{ is odd} \\ \frac{k!}{(\frac{k}{2})!(\frac{k}{2}+1)!} & \end{cases} = C_k$

(Catalan's number).

<u>Hint</u> Eg $\longleftrightarrow$ arrangements of brackets ((()())(()))()
"(": new vertex, ")": back to prev. vertex.

Hence $\int x^k \, \mathbb{E} d\mu_N \longrightarrow \frac{C_k}{2^k} \underset{\uparrow}{=} \int x^k d\mu_W(x)$.

elementary computation by induction

(ii): similar comb. argument for pairs of paths (the contribution to

$$\mathbb{E}\left[\frac{1}{N} \operatorname{tr}(H/2\sqrt{N})^k \cdot \frac{1}{N} \operatorname{tr}(H/2\sqrt{N})^{\ell}\right]$$

comes from pairs of disjoint trees.)

B (Kesten ; McKay)

Consider a sequence of graphs $\{G_N = (V_N, E_N)\}$, $\# V_N = N$.
~~Denote~~ Assume that

- G_N is d-regular for any N
- $\text{girth}(G_N) = \underset{\text{shortest cycle}}{\text{length of}} \xrightarrow[N\to\infty]{} \infty$

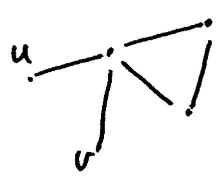
Denote by μ_N the spectral measure of $A_N / 2\sqrt{d-1}$, where A_N is the adjacency matrix of G_N.

Thm $\mu_N \xrightarrow[N\to\infty]{w} \mu_{KM}^{d}$.

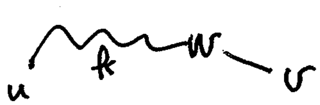
One can use moments as before and show that only trees contribute. Let us show a modification of this argument.

Let $B_{k,N}$ be an $N \times N$ matrix,

$$B_{k,N}(u,v) = \# \begin{array}{l}\text{paths of length } k \\ \text{from } u \text{ to } v \text{ in } G_N \\ \text{without backtracking}\end{array}$$

(eg  is OK, but not ).

Lemma $B_{k,N} \cdot A_N = B_{k+1,N} + (d-1) B_{k-1,N}$

Proof 
bt on last step $\longrightarrow B_{k-1,N}$
no bt $\longrightarrow B_{k+1,N}$

▨

Therefore $B_{k,N}$ is a polynomial of A_N. Observe that, for every $k>0$, $\mathrm{tr}\, B_{k,N}$ is eventually zero.

Now we are almost done:

$$\frac{B_{k,N}}{\sqrt{d(d-1)^{k-1}}} = P_k\left(A_N / 2\sqrt{d-1}\right), \quad \text{and}$$

$$2x P_k(x) = P_{k+1}(x) + P_{k-1}(x).$$

Checking the cases $k=0,1,2$, one may see that these are exactly the orthogonal pol-s w.r.t. to μ^d_{KM}. Thus:

$$\frac{1}{N} \mathrm{tr} \frac{B_{k,N}}{\sqrt{d(d-1)^{k-1}}} = \int P_k \, d\mu_N \longrightarrow 0 =$$

$$= \int P_k \cdot 1 \, d\mu^d_{KM} \quad , k = 1, 2, \ldots$$

therefore $\mu_N \longrightarrow \mu^d_{K,M}$.

<u>Rmks</u> 1) One can put ± 1-s on the edges
↳ "weighted" adj. matrix

2) also, one may allow cycles, as long as there are "not many" of them.
EG: a <u>random</u> d-regular graph is fine.

3) The argument can be also used to prove Wigner's thm. this is esp. easy for

$$H_N(u,v) = \begin{cases} \pm 1, & u<v \\ 0, & u=v \end{cases}.$$

4) (unlike moments), one can deduce a reasonable estimate on the rate of convergence.