Last time

H_N - $N\times N$ Hermitian matrix

global regime: $\frac{\#\{\lambda_i \in B\}}{N} \sim ?$

$\mu_{H_N} = \frac{1}{N}\sum \delta_{\lambda_i}$, then this is $\mu_{H_N}(B)$.

Wigner's thm

H_N - matrix with indep. entries above the diagonal

$\mathbb{E} H_N(u,v) = 0$, $\mathbb{E} H_N(u,v)^2 = 1$ + technical cond-s

μ_N - the spectral measure of $H_N/2\sqrt{N}$

Then $\mu_N \longrightarrow \mu_W$, $\frac{d\mu_W}{dx} = \frac{2}{\pi}(1-x^2)_+^{1/2}$

We have proved a bit less:

$\mathbb{E}\mu_N \longrightarrow \mu_W$, $\mathbb{E}\mu_N(B) = \mathbb{E}\frac{\#\{\lambda_i \in B\}}{N}$.

Idea: $\int x^k \, d\mathbb{E}\mu_N \longrightarrow \int x^k \, d\mu_W$,

contribution comes from trees.

Before leaving the global regime,

1) another theorem
2) another method of proof

and to save time: another thm. using another method of proof.

Consider a sequence of graphs, $\{G_N = (V_N, E_N)\}$, $\#V_N = N$.
Assume that

1) G_N is d-regular
2) $\text{girth}(G_N) = \underset{\text{shortest cycle}}{\text{length of}} \xrightarrow[N\to\infty]{} \infty$.

Let μ_N be the spectral measure of $A_N/2\sqrt{d-1}$, where

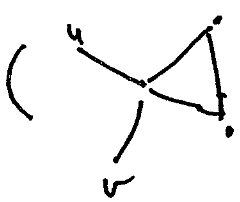
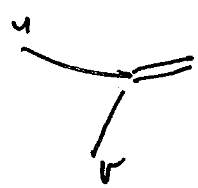
$$A_N(u,v) = \begin{cases} 1, & u \sim v \\ 0 \end{cases}$$

<u>Thm</u> $\mu_N \xrightarrow{w} \mu^d_{KM}$, $\dfrac{d\mu^d_{KM}}{dx} = \dfrac{2d(d-1)}{\pi} \dfrac{\sqrt{1-x^2}}{d^2 - 4(d-1)x^2}$

One can use moments (again, only trees contribute).
Let us show a modification of this argument.

Let $B_{k,N}$ be an $N \times N$ matrix,

$B_{k,N}(u,v) = \#$ paths of length k from u to v in G_N without backtracking $(u_{j+2} \neq u_j)$

( is OK, but not )

<u>Lemma</u> $B_{k,N} \cdot A_N = B_{k+1,N} + (d-1) B_{k-1,N}$

<u>Proof</u> $u \overset{k}{\sim\sim\sim} w \text{---} v$
- bt on last step $\rightsquigarrow B_{k-1,N}$
- no bt on last step $\rightsquigarrow B_{k+1,N}$.

$\blacksquare$

Therefore $B_{k,N} = q_k(A_N)$,

$$x\, q_k(x) = q_{k+1}(x) + (d-1)\, q_{k-1}(x).$$

Thus $\dfrac{B_{k,N}}{\sqrt{d(d-1)^{k-1}}} = P_k\left(A_N/2\sqrt{d-1}\right)$, and

$$2x\, P_k(x) = P_{k+1}(x) + P_{k-1}(x).$$

Checking $k=0,1,2$, we see that these are exactly the orth. pol-s with resp. to μ_{KM}^{d}, that we have seen before.

Thus $\underset{N\gg 1}{\overset{\uparrow}{0}} = \dfrac{1}{N} \operatorname{tr} \dfrac{B_{k,N}}{\sqrt{d(d-1)^{k-1}}} = \int P_k \, d\mu_N$, and

on the other hand

$$0 = \int P_k \cdot 1 \, d\mu_{KM}^{d}, \quad k = 1,2,\ldots$$

Thus $\mu_N \longrightarrow \mu_{KM}^{d}$.

$\blacksquare$

<u>Rmks</u> 1) one can put $\pm$-s on edges

2) allow cycles, as long as there are not too many of them (a <u>random</u> d-reg graph is ok)

3) estimate on the rate of convergence

4) argument can be used to prove Wigner's thm (esp easy for $H_N(u,v) = \begin{cases} \pm 1, & u \neq v \\ 0 \end{cases}$)

) The edge of the spectrum

According to Wigner's law,

$$\#\{\lambda_i \in B\} \approx N \int_B \frac{2}{\pi}(1-x^2)_+^{1/2}\,dx. \qquad (+)$$

Is this also true for "small" B? We focus on B near the edge (e.g. $B = (1+b_1, 1+b_2)$).

Let $B = (1-\varepsilon, 1)$. Then

$$N \int_B \frac{2}{\pi}(1-x^2)_+^{1/2}\,dx \asymp N\varepsilon^{3/2},$$

so, for $\varepsilon \asymp N^{-2/3}$, the RHS is $\Theta(1)$ (and not necessarily integer), whereas the LHS of (+) is integer. So (+) can not hold on the scale $N^{-2/3}$.

It appears that this is the only obstruction: (+) holds on any larger scale (eg $N^{-2/3}\log\log N$). The formal def-s: below.

We focus on the average:

$$\mathbb{E}\#\{\lambda_i \in B\};$$

the same methods can be used to solve the (more interesting) q-s about gap distribution, et cet (eg the distribution of $\frac{\lambda_{max} - \mathbb{E}\lambda_{max}}{\sqrt{\mathrm{Var}\,\lambda_{max}}}$).

<u>Def</u> the Airy f-n is a (special) solution of $y'' = xy$ (ie $Ai''(x) = x\,Ai(x)$). $\left[Ai(x) \sim \frac{e^{-\frac{2}{3}x^{3/2}}}{2\sqrt{\pi}\,x^{1/4}},\ x \to +\infty \right.$

<u>Goal</u> $\mathbb{E}\,\#$ eigenvalues in $1 + (2N^{2/3})^{-1} A \longrightarrow$

$$\longrightarrow \int_A \left[Ai'(x)^2 - x\,Ai(x)^2 \right] dx$$

(say, for A being a union of intervals, bounded from below)

<u>Soshnikov '97</u>: the statement holds under the following assumptions:

(i) H_N is Hermitian, $\{ H_N(u,v) \mid u \le v \}$ are independent

(ii) $\mathbb{E}\,H_N(u,v) = 0$; $H_N(u,v)$ is "isotropic" for $u<v$

(iii) $\mathbb{E}\,|H_N(u,v)|^{2k} \le (Ck)^k$ ("subgaussian tails")

<u>Rmk</u> Earlier proved for one (important) special distribution (GUE): Forrester, Tracy–Widom, ...

[Hence it is sufficient to prove that the limit does not depend on the R.M.]

We focus on the special case:

$$H_N(u,v) \begin{cases} \sim U(\mathbb{T}), & u<v \\ 0, & u=v. \end{cases}$$

How do moments help?

Let $\mu_i = (\lambda_i - 1)\, 2N^{2/3}$, we are interested in $\lim \mathbb{E}\xi_N(A)$, $\xi_N = \sum \delta_{\mu_i}$.

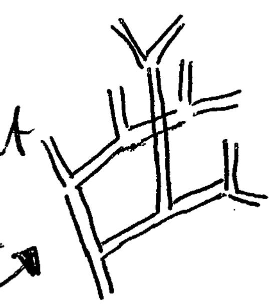
$$\mathbb{E}\, \mathrm{tr}\left(\frac{H_N}{2\sqrt{N}}\right)^m = \mathbb{E} \sum \lambda_i^m = \mathbb{E} \sum \left(1 + \frac{\mu_i}{2N^{2/3}}\right)^m \simeq$$

$$\simeq \mathbb{E} \sum \exp\left(\frac{m}{2N^{2/3}}\mu_i\right) = \int \exp \frac{m\mu}{2N^{2/3}}\, d\mathbb{E}\xi_N .$$

easy to justify

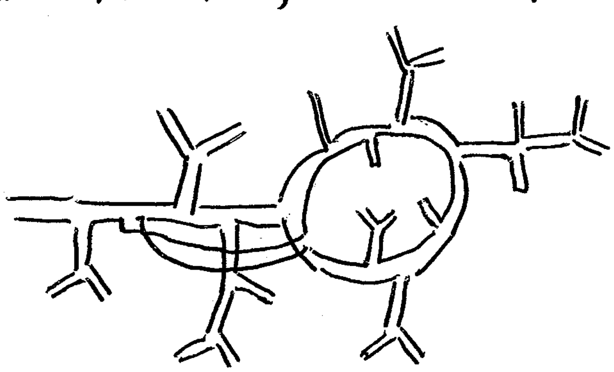
Setting $m \asymp 2\alpha N^{2/3}$, we obtain the Laplace transform of $\mathbb{E}\xi_N$ at the point α. Hence we need to study moments of order $\Theta(N^{2/3})$ (as opposed to $\Theta(1)$, as before).

Sinai & Soshnikov: if $m \ll N^{2/3}$, the dominant paths are still trees (appropriately defined).



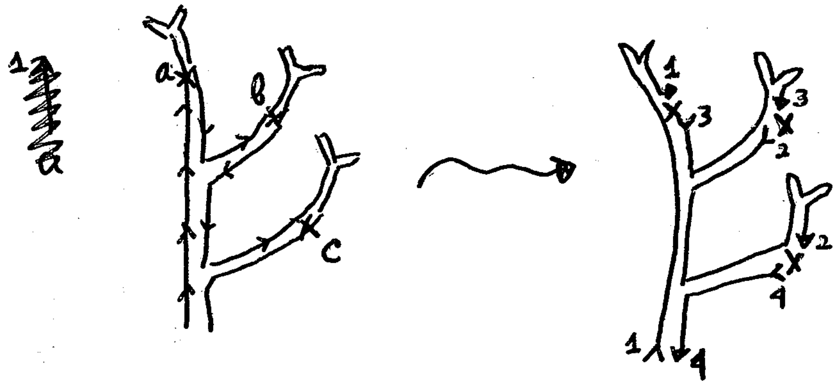
($\Leftrightarrow$ for $s_N \ll N^{2/3}$, nothing interesting happens).

For $m \asymp N^{2/3}$, other paths appear, e.g.:



Soshnikov's approach:

pick a tree, and do the following g ($\in \mathbb{Z}_+$) times: choose 3 vertices a, b, c on the tree, and glue them as follows:



(now $a=b=c$)

[The picture on the prev. page corresponds to $g=1$].

<u>Fact</u> "all" paths are obtained in this way, thus

$$\# \text{paths}(2m) =$$

$$= (1+o(1)) \cdot \underbrace{\#\text{trees}}_{\substack{\text{computed}\\ \text{via Catalan}\\ \text{numbers}}} \cdot \sum_{g \geq 0} \left[\frac{(m+1)^3}{2}\right]^g / g!$$

[valid for $m \ll N^{0.99}$; in particular: $m = \alpha N^{2/3}$]

<u>Exercise</u> Find the limit, and show that it satisfies the same differential equation as

$$\varphi(\alpha) = \int e^{\alpha x} \left[Ai'(x)^2 - x\, Ai(x)^2 \right] dx.$$