Prev. work:

$H_N$ – Hermitian $N \times N$ random matrix,

for simplicity: $H_N(u,v) \begin{cases} \sim U(\mathbb{T}), & u<v \\ 0, & u=v \end{cases}$

~~We~~ Soshnikov:

$$\mathbb{E}\ \#\text{ eigenvalues of } \frac{H_N}{2\sqrt{N}} \text{ in } 1+(2N^{2/3})^{-1}A$$

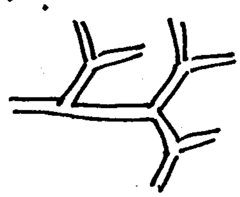
$$\xrightarrow[N\to\infty]{} \int_A \left[Ai'(x)^2 - x\,Ai(x)^2\right] dx,$$

for reasonable $A$ (eg finite union of intervals, bounded from below).

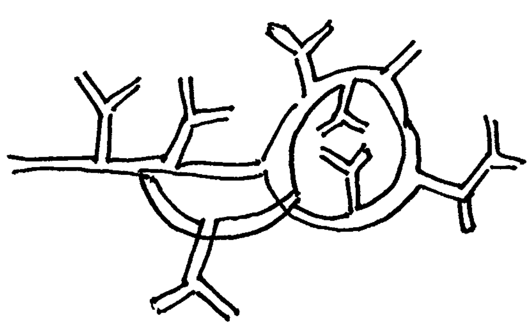
Main step: study $\mathbb{E}\,\mathrm{tr}\,H_N^m$ for $m \asymp N^{2/3}$

$$\mathbb{E}\,\mathrm{tr}\,H_N^m = \sum \underbrace{\mathbb{E}\,H_N(u_0,u_1)\cdots H_N(u_{m-1},u_0)}_{=\begin{cases}1, & u_0u_1\cdots u_0 \text{ is a path passing every edge the same nr of times in both directions} \\ 0 & \neg\end{cases}}$$

for $m \ll N^{2/3}$:

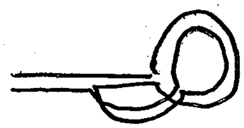
trees  are dominant

$m \asymp N^{2/3}$: other paths appear, eg



Soshnikov: classifications ~~in~~ of these paths.

Feldheim–S:

the picture above is  + trees glued onto it.

Erase all trees, compute everything, and then glue them back.

Analytically: corresponding to $\mu^d_{KM}$ with $d = N-1$

$\mathbb{E}\, \mathrm{tr}\, P_n\left(\frac{H_N}{2\sqrt{N-2}}\right)$ counts paths without backtracking (= without trees),

and "converges" to

$$\int_{-\infty}^{\infty} \frac{\sin \frac{n\sqrt{\mu}}{N^{1/3}}}{\sqrt{\mu}}\, d\, \mathbb{E}\xi_N(\mu).$$

Taking $n \asymp \alpha N^{1/3}$ (not $2/3$ !), we obtain another transform of $\mathbb{E}\xi_N$:

$$\int \frac{\sin \alpha\sqrt{\mu}}{\alpha\sqrt{\mu}}\, d\, \mathbb{E}\xi_N(\mu)$$

Strange, but invertible (sometimes): [B.M.Levitan '50]

So we need to count n.b.t. paths of length $\Theta(N^{1/3})$.

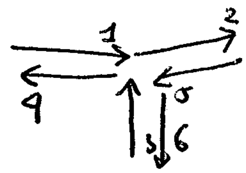
Paths without backtracking:

only vertices of degree $\underline{\underline{3}}$ matter

degree 2: just go forward / backward

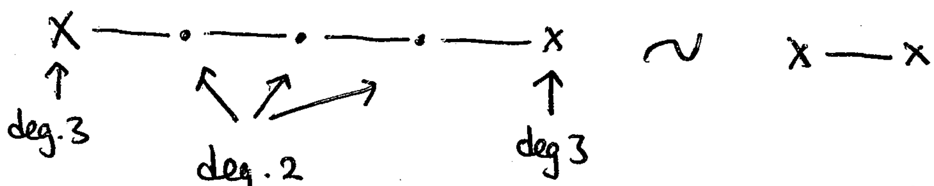
degree >3: (asymptotically) negligible.

degree 3:

e.g.: 

(et cet.)

Equivalence classes of paths:

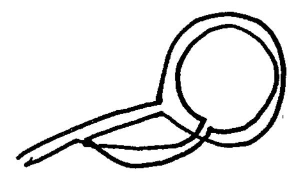



<u>Main fact</u> $\exists\, g \geq 1$, so that

$\#$ vertices of degree 3 $= 4g$
(+root)

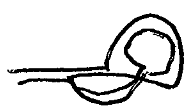
$\#$ (long) edges $= 6g-1$

The number of equivalence classes ~~satisfies~~ corresponding to given $g \geq 1$ satisfies



$g=1$
$\bar{E}=5$
$\bar{V}=4$

$$(g/c)^{2g} \leq D(g) \leq (Cg)^{2g}.$$

Equiv. class  $\rightsquigarrow \sim \frac{n^4}{4!}$ paths

↳ # ways to divide $n$ into 5 pieces

Thus finally:

$$(1+o(1)) \sum D(g) \cdot \frac{n^{6g-2}}{(6g-2)!} \; N^{n+1-2g} \qquad (n \ll N^{0.99})$$

④ Summary of the argument:

- paths are grouped into equivalence classes
- $\#\,\text{paths} = (1+o(1)) \cdot \sum_{g\geq 1} \underbrace{D(g)}_{\substack{\text{combinatorial}\\ \text{coef.}}} \cdot \frac{n^{6g-2}}{(6g-2)!} N^{n+1-2g}$

The estimate is true for $n \ll N^{0.99}$, but we only need it for $n = \Theta(N^{1/3})$.

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This was a "mean-field" model. One can think of it as follows:

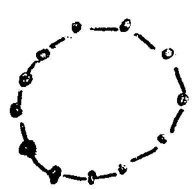
$H_N(u,v)$ – "random noise" on every edge of $K_N$

Now consider a particle moving on $K_N$

classical particle $\rightsquigarrow$ Random Walk in Random Environment

quantum particle: $e^{itH_N}$ is the evolution operator.

A more physical model: start from a different graph, with some underlying geometry.

The simplest example: 

but then – no RM behaviour ($H_N$ is 3-diagonal, so everything can be computed).

How to add a parameter?

## Band matrices

$\mathcal{G}_N = (\mathbb{Z}/N\mathbb{Z},\ \mathcal{E}_N)$ $\qquad u \sim v \Leftrightarrow 0 < |u-v|_N \le W_N,$

where $|u|_N = \min(|u|, N-|u|)$,

$W_N$ - a parameter

$$H_N(u,v) \begin{cases} \sim U(\mathbb{T}), & u \sim v \\ 0 \end{cases}$$

**Thm** $\mathbb{E}\,\#$ eigenvalues of $\dfrac{H_N}{2\sqrt{2W_N-1}}$ in $1 + (2N^{2/3})^{-1}A$

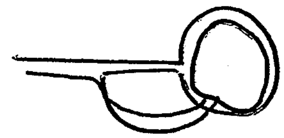
$$\longrightarrow \int_A \left[Ai'(x)^2 - x\,Ai(x)^2\right] dx$$

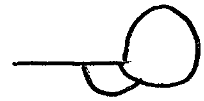
iff $\dfrac{W_N}{W^{5/6}} \longrightarrow +\infty.$

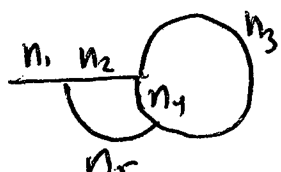
In fact, if $\dfrac{W_N}{N^{5/6}} \to \infty$, the whole distribution of $\#\ldots$ tends to the same limit as for Wigner matrices.

Proof 1) $\mathbb{E}\,\mathrm{tr}\,P_n\left(\dfrac{H_N}{2\sqrt{2W_N-1}}\right)$ counts non-backtracking walks on $\mathcal{G}_N$ that pass every edge the same number of times in both directions.

As before, we divide these paths into equivalence classes.

Q-n How many paths are there in a given equivalence class? Eg  ?

= # ways to "embed"  into $G_N$ so that the lengths of the edges sum up to $n$

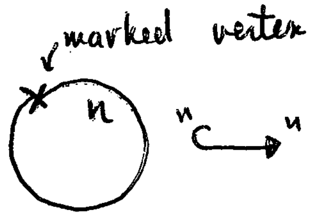



$n_1 + n_2 + n_3 + n_4 + n_5 = n \quad (= \Theta(N^{1/2}))$

For $K_N$ this was trivial:

$$(1+o(1)): \quad \frac{n^4}{4!} \cdot N^{n-1} = N^{n+1}(1+o(1)) \cdot \frac{n^4}{4!} \cdot \frac{1}{N^2}$$

goal: get the same, up to scaling.

Even simpler q-n: 

"$\hookrightarrow$" $G_N$

(in $K_N$: $N^n$)

In $G$: $\underset{\substack{\uparrow \\ \text{where to start} \\ \text{(marked vertex)}}}{N} \cdot \underset{\substack{\uparrow \\ \text{where to go} \\ \text{on 1st step}}}{(2W)} \cdot (2W-1)^{n-1} \cdot \mathbb{P}\left\{\begin{array}{l}\text{return to } x, \\ \text{and not from} \\ \text{the same side}\end{array}\right\}$

$\| \leftarrow$ assuming perfect mixing

$$\frac{1}{N} \cdot \frac{2W-1}{2W}$$

thus, assuming perfect mixing, we get:

$$(2W-1)^n \quad (\text{vs } N^n \text{ for } K_N)$$

ie, the same, up to scaling. When does this work?

$n = \Theta(N^{1/3})$, hence we need

$$\underbrace{T_{mix}}_{\substack{\text{mixing time} \\ \text{of RW on } \mathcal{G}}} \ll N^{1/3}.$$

[strictly speaking, non-backtracking RW, but the difference is not essential here

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<u>Reminder</u> starting from a distribution $p_0$ on $\mathcal{G}$, the distribution after $n$ steps is

$$\frac{\mathcal{A}^n}{d^n} p_0, \text{ where } \mathcal{A} \text{ is the adj. matrix of } \mathcal{G}, \ d = 2W.$$

Decompose: $p_0 = \underbrace{p_{stat}}_{\text{uniform}} + \underbrace{p^\perp}_{\text{orthogonal to } p_{stat}}$, then

$$\left\| \frac{\mathcal{A}^n}{d^n} p_0 - p_{stat} \right\| \leq \lambda_2^n \| p^\perp \|,$$

where $\lambda_2$ is the second largest eigenvalue of $\mathcal{A}/d$ (or rather, $\max(|\lambda_2|, \dots, |\lambda_N|)$).

Thus the mixing time is of order $(1-\lambda_2)^{-1}$.

Now, $\frac{\mathcal{A}}{d}$ commutes with cyclic shifts, hence the Fourier transform (mod $N$) diagonalises it. One easily finds:

$$1 - \lambda_2 = \Theta\left(\frac{W^2}{N^2}\right) \Rightarrow T_{mix} \asymp \frac{N^2}{W^2}.$$

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Going back: $\frac{N^2}{W^2} \ll N^{1/3} \Leftrightarrow W \gg N^{5/6}$.

Fyodorov & Mirlin ['91–'94] have suggested the following heuristic, based on the Thouless criterion ['72]:

RM behaviour of eigenvalues near $\lambda_0$ $\iff$ $T_{mix} \ll \dfrac{\langle \text{density at } \lambda_0 \rangle}{\langle \text{spacing at } \lambda_0 \rangle}$

Thus for $-1 < \lambda_0 < 1$

$$\text{RM behaviour} \overset{?}{\iff} \frac{N^2}{W^2} \ll \frac{N}{1} \iff W \gg \sqrt{N}$$

On the physical level of rigour, this has been justified and elaborated by F-M.

<u>Major problem</u>: justify this (mathematically)

For $\lambda_0 = 1$, F-M suggest

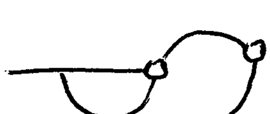
$$\text{RM behaviour} \overset{?}{\iff} \frac{N^2}{W^2} \ll \frac{N^{2/3}}{N^{1/3}} \iff W \gg N^{5/6}$$

This is what we have proved.

<u>Rmk</u> We embed more general subgraphs, eg

, edge by edge

then one needs to make sure that all the edges are of length $N^{1/3}$, not:



In our case this follows from the fact that $\mathcal{G}$ has few short cycles (this is not a mixing condition). So