

A stability result for mean width of L_p -centroid bodies.

B. Fleury*, O. Guédon* and G. Paouris†

Abstract

We give a different proof of a recent result of Klartag [12] concerning the concentration of the volume of a convex body within a thin Euclidean shell and proving a conjecture of Anttila, Ball and Perissinaki [1]. It is based on the study of the L_p -centroid bodies. We prove an almost isometric reverse Hölder inequality for their mean width and a refined form of a stability result.

1 Introduction

In this paper we study how the volume of a symmetric convex body concentrates within a very thin Euclidean shell. Let K be an isotropic convex body in \mathbb{R}^n i.e. a symmetric convex body of volume 1 such that for some fixed $L_K > 0$,

$$\forall \theta \in S^{n-1}, \int_K \langle x, \theta \rangle^2 dx = L_K^2.$$

It is known that every symmetric convex body has an affine image which is isotropic. We denote by $|x|_2$ the Euclidean norm of $x \in \mathbb{R}^n$. In the paper [1], Anttila, Ball and Perissinaki asked if every isotropic convex body in \mathbb{R}^n satisfy an ε -concentration hypothesis namely:

Concentration hypothesis. Does there exist ε_n such that $\lim_{n \rightarrow \infty} \varepsilon_n = 0$ and

$$\left| \left\{ x \in K, \left| \frac{|x|_2}{L_K \sqrt{n}} - 1 \right| \geq \varepsilon_n \right\} \right| \leq \varepsilon_n?$$

We will prove the following

Theorem 1. *There exists c and c' such that for every isotropic convex body K in \mathbb{R}^n , and every $p \leq (\log n)^{1/3}$,*

$$1 \leq \left(\int_K |x|_2^p dx \right)^{1/p} / \left(\int_K |x|_2 dx \right) \leq 1 + c p / (\log n)^{1/3}.$$

In particular, for every $\varepsilon \in (0, 1)$,

$$\left| \left\{ x \in K, \left| \frac{|x|_2}{\sqrt{n} L_K} - 1 \right| \geq \varepsilon \right\} \right| \leq 2e^{-c\sqrt{\varepsilon}(\log n)^{1/12}}. \quad (1)$$

*Université Pierre et Marie Curie, Institut de Mathématiques de Jussieu, boîte 186, 4 Place Jussieu, 75252 Paris Cedex 05, FRANCE. *E-mail:* fleury_bruno@yahoo.fr, guedon@math.jussieu.fr

†Université de Marne-la-Vallée, Laboratoire d'Analyse et de Mathématiques Appliquées, 5 Bd Descartes, Champs-sur-Marne, 77454 Marne-la-Vallée Cedex 2, FRANCE. *E-mail:* grigoris.paouris@yahoo.co.uk Research supported by a Marie Curie Intra-European Fellowship (EIF), Contract MEIF-CT-2005-025017.

This implies that the concentration hypothesis holds with $\varepsilon_n = c(\log \log n)^2/(\log n)^{1/6}$. This result has been very recently obtained in full generality by Klartag [12], where he proved that (1) holds true with $2e^{-\varepsilon^2 \log n}$ for every isotropic convex body with center of mass at the origin. Our goal is to present a different approach via the notion of L_p -centroid bodies. To any star shape body with respect to the origin, $L \subset \mathbb{R}^n$, we associate its L_p -centroid body $Z_p(L)$ which is a symmetric convex body defined by its support function:

$$\forall y \in \mathbb{R}^n, h_{Z_p(L)}(y) = \left(\int_L |\langle x, y \rangle|^p dx \right)^{1/p}.$$

This body is an homothetic of the L_p -centroid body defined by Lutwak and Zhang in [16] (see also [15]). For any symmetric convex body C , we define the p -th mean width as

$$W_p(C) = \left(\int_{S^{n-1}} h_C(\theta)^p d\sigma(\theta) \right)^{1/p}.$$

The main result of this paper compares the mean width of the L_p -centroid bodies of an isotropic convex body to the mean width of the L_p -centroid bodies of the Euclidean unit ball of volume 1.

Theorem 2. *There exists a constant c such that for any n , for every isotropic convex body K in \mathbb{R}^n , if \tilde{D} denotes the Euclidean unit ball in \mathbb{R}^n of volume 1, for every $p \leq (\log n)^{1/3}$*

$$\frac{W_1(Z_p(K)) W_1(Z_1(\tilde{D}))}{W_1(Z_1(K)) W_1(Z_p(\tilde{D}))} \leq 1 + cp/(\log n)^{1/3}.$$

Regarding K as a probability space, these techniques were used by the third named author [20] to prove that the L_q -norms of the Euclidean norm are almost constant for any $q \leq \sqrt{n}$, i.e. (see theorem 1.2 in [20])

$$\exists C \geq 1, \forall q \leq c\sqrt{n}, \left(\int_K |x|_2^q dx \right)^{1/q} \leq C \left(\int_K |x|_2^2 dx \right)^{1/2} = C\sqrt{n} L_K. \quad (2)$$

Theorem 1 is in fact an almost isometric version of this result (although it does not recover the full isomorphic one). It is also related to a weak form of Kannan, Lovász and Simonovits [11] conjecture about Cheeger-type isoperimetric constant for convex bodies: does there exist $c > 0$ such that for any isotropic convex body K ,

$$\sigma_K^2 := \frac{\text{Var}(|X|_2^2)}{nL_K^4} \leq c$$

where X is a random vector uniformly distributed on K ? We refer to the paper of Bobkov [4] for more details between the full KLS-conjecture and this weaker form. Theorem 1 implies that $\lim_{n \rightarrow \infty} \sigma_K^2/n = 0$. Up to now, the only known upper bound was the trivial one, $\sigma_K \leq c\sqrt{n}$.

On the way, we will need a new type of stability result for the L_p -centroid bodies. Let K and L be symmetric convex bodies of volume 1 in \mathbb{R}^d , if $Z_p(L)$ is close to $Z_p(K)$ for the geometric distance, what can we say about the geometric distance between K and L ? This type of question has been studied by Bourgain and Lindenstrauss [6] in the case of projection bodies i.e. $p = 1$. We will prove a more precise result when one of the bodies is the Euclidean unit ball D . The geometric distance between two symmetric convex bodies K and L is defined by

$$d(K, L) = \inf \{ab \mid a, b > 0 \text{ and } 1/aK \subset L \subset bK\}.$$

Theorem 3. *There exists $c > 0$ such that for every integer d greater than 3 and any odd integer $p \leq d$, if K is a symmetric convex body in \mathbb{R}^d such that for some $\alpha > 1$ and $\varepsilon \in (0, (c\alpha)^{-2d^3})$*

$$d(K, D) \leq \alpha \quad \text{and} \quad d(Z_p(\tilde{D}), Z_p(\tilde{K})) \leq 1 + \varepsilon$$

where $\tilde{K} = |K|^{-1/d}K$ and $\tilde{D} = |D|^{-1/d}D$ then

$$d(K, D) \leq 1 + h(\varepsilon) \quad \text{and} \quad (1 - h(\varepsilon))Z_p(\tilde{D}) \subset Z_p(\tilde{K}) \subset (1 + h(\varepsilon))Z_p(\tilde{D})$$

where $h(\varepsilon) = (c\alpha)^{d+p+1}\varepsilon^{1/d^2}$.

It was proved in [1] that the concentration hypothesis implies some type of central limit theorem. The conjecture about a central limit theorem for convex sets stated by Anttila, Ball, Perissinaki [1] and Brehm, Voigt [7] has been recently proved by Klartag [12] and we refer to that paper for more precise references on this subject.

The paper is organized as follows. In Section 2, we will explain how we reduce the study of concentration of the volume of an isotropic convex body to the study of its L_p -centroid bodies. We will prove the main Theorem 2 in Section 3. The proof of Theorem 3 is done in Section 4 and uses standard tools coming from the theory of spherical harmonics.

Notations. Throughout this paper, D will be the Euclidean ball in \mathbb{R}^n and S^{n-1} the unit sphere. The volume is denoted by $|\cdot|$. We write ω_n for the volume of D and σ for the rotationally invariant probability measure on S^{n-1} . Also we write \tilde{L} for the homothetic image of volume 1 of the body $L \subset \mathbb{R}^n$, that is $\tilde{L} = |L|^{-1/n}L$ and $R(L)$ will be the circumradius of L i.e. the smallest real number such that $L \subset R(L)D$. The letter c will always be used as being a universal constant and it can change from line to line.

Acknowledgement. The authors would like to thank K. Ball for several useful discussions.

2 Reduction to L_p centroid bodies.

For any isotropic convex body K , we define $I_p(K) = (\int_K |x|_2^p dx)^{1/p}$. It is easy to check that there exists a constant $c_{n,p}$ such that for every $\theta \in S^{n-1}$

$$c_{n,p}^p \int_{S^{n-1}} |\langle \theta, x \rangle|^p d\sigma(\theta) = |x|_2^p, \quad \text{i.e.} \quad c_{n,p} = \left(\frac{\sqrt{\pi} \Gamma(\frac{p+n}{2})}{\Gamma(\frac{p+1}{2}) \Gamma(\frac{n}{2})} \right)^{1/p}.$$

Note that $c_{n,p}$ is similar to $\sqrt{(n+p)/p}$. By Fubini theorem and the definition of $W_p(Z_p(K))$, $I_p(K) = c_{n,p} W_p(Z_p(K))$. We first need some precise computations in the case of the Euclidean ball of volume 1.

Lemma 1. *Let D be the Euclidean unit ball in \mathbb{R}^n , then for any $p \leq n$,*

$$I_p(\tilde{D})/I_1(\tilde{D}) \leq 1 + cp/n^2. \quad (3)$$

Let k be an integer and $p \leq k \leq n$ and denote by \widetilde{D}_F the Euclidean unit ball of volume 1 in any k -dimensional subspace F of \mathbb{R}^n then

$$\left(W_1(Z_1(\tilde{D}))/W_1(Z_p(\tilde{D})) \right) \left(W_1(Z_p(\widetilde{D}_F))/W_1(Z_1(\widetilde{D}_F)) \right) \leq 1 + c p/k.$$

Proof. For any $1 \leq p \leq n$, we have

$$\frac{c_{n,p} W_p(Z_p(\tilde{D}))}{c_{n,1} W_1(Z_1(\tilde{D}))} = \left(\int_{\tilde{D}} |x|_2^p dx \right)^{1/p} / \int_{\tilde{D}} |x|_2 dx = (1 + 1/n)(1 + p/n)^{-1/p} \leq 1 + cp/n^2.$$

Since for any $p \geq 1$, $W_1(Z_p(\tilde{D})) = W_p(Z_p(\tilde{D}))$ and $x\Gamma(x) = \Gamma(x+1)$, we get

$$\frac{W_1(Z_1(\tilde{D})) W_1(Z_p(\tilde{D}_F))}{W_1(Z_p(\tilde{D})) W_1(Z_1(\tilde{D}_F))} = \left(\frac{\Gamma(1 + \frac{n+p}{2}) \Gamma(1 + \frac{k}{2})}{\Gamma(1 + \frac{n}{2}) \Gamma(1 + \frac{k+p}{2})} \right)^{1/p} \frac{\Gamma(1 + \frac{n}{2}) \Gamma(1 + \frac{k+1}{2})}{\Gamma(1 + \frac{n+1}{2}) \Gamma(1 + \frac{k}{2})}.$$

Simple computations with the Γ function proves the stated estimate when $p \leq k$. \square

For any fixed symmetric convex body L , Litvak, Milman and Schechtman [14] studied the behavior of $W_p(L)$ as a function of p .

Lemma [14] *Let L be a symmetric convex body of \mathbb{R}^n then for any $p \leq c_1 n (W_1(L)/R(L))^2$,*

$$|W_p(L) - W_1(L)| \leq \|h_L(u) - W_1(L)\|_p \leq c_2 \sqrt{\frac{p}{n}} R(L) \quad (4)$$

where c_1 and c_2 are universal constants.

The next lemma was essentially proved in [20].

Lemma 2. *There exists $c > 0$ such that for every isotropic convex body $K \subset \mathbb{R}^n$, for every $1 \leq p \leq c\sqrt{n}$,*

$$R(Z_p(K)) \leq c\sqrt{p} W_1(Z_p(K)). \quad (5)$$

Proof. We briefly indicate a proof. In isotropic position, $R(Z_p(K)) \leq cpR(Z_2(K)) = cpL_K$. Corollary 3.11 in [20] means that if $p \leq c\sqrt{n}$, $W_p(Z_p(K))$ is similar up to universal constants to $W_1(Z_p(K))$. Observe that $W_p(Z_p(K)) \geq c\sqrt{p/n}I_p(K) \geq c\sqrt{p}L_K$ and $\sqrt{p}W_1(Z_p(K)) \geq cpL_K \geq cR(Z_p(K))$. \square

Proof of Theorem 1. We write

$$\frac{I_p(K)I_1(\tilde{D})}{I_1(K)I_p(\tilde{D})} = \frac{W_p(Z_p(K))}{W_1(Z_p(K))} \left(\frac{W_1(Z_p(K)) W_1(Z_1(\tilde{D}))}{W_1(Z_1(K)) W_1(Z_p(\tilde{D}))} \right). \quad (6)$$

From (4) and (5), we get $1 \leq W_p(Z_p(K))/W_1(Z_p(K)) \leq 1 + c \frac{p}{\sqrt{n}}$ when $p \leq c\sqrt{n}$. Hence Theorem 1 is proved using (3), (6) and Theorem 2. In particular,

$$\int_K \left(\frac{|x|_2^2}{nL_K^2} - 1 \right)^2 dx = \frac{I_4(K)}{I_2^4(K)} - 1 \leq c/(\log n)^{1/3}. \quad (7)$$

The function $f(x) = \left(\frac{|x|_2^2}{nL_K^2} - 1 \right)$ is a polynomial of degree 2 and we can use the results of Bobkov [3] about L_r -norms of polynomials. Indeed, theorem 1 of [3] states that there exists a universal constant $c > 0$ such that $\int_K e^{\hat{f}(x)/c} \int_K \hat{f}(x) dx \leq 2$ where $\hat{f} = |f|^{1/2}$. For every $\varepsilon \in (0, 1)$, since $\int_K \hat{f}(x) dx \leq (\int_K f^2(x) dx)^{1/4}$, we get by (7) and by Chebychev inequality

$$\left| \left\{ x \in K, \left| \frac{|x|_2}{\sqrt{n}L_K} - 1 \right| \geq \varepsilon \right\} \right| \leq \left| \left\{ x \in K, \left| \frac{|x|_2^2}{nL_K^2} - 1 \right| \geq \varepsilon \right\} \right| \leq 2e^{-c\sqrt{\varepsilon}(\log n)^{1/12}}.$$

\square

3 Proof of Theorem 2

We now introduce some notations and recall some well known facts from local theory of Banach spaces. The subspace $F \subset \mathbb{R}^n$ being chosen, denote by E the orthogonal subspace of F and for every $\phi \in S_F$, the Euclidean sphere in F , we define $E(\phi)$ to be $\{x \in \text{span}\{E, \phi\}, \langle x, \phi \rangle \geq 0\}$. For any $q \geq 0$, define the star body B_q by its radial function

$$\forall \phi \in S_F, r_{B_q}(\phi) = \left(\int_{K \cap E(\phi)} |\langle x, \phi \rangle|^q dx \right)^{1/(q+1)}.$$

A theorem of Ball [2] asserts that when K is a symmetric convex body in \mathbb{R}^n , this radial function defines a symmetric convex body in F . These balls are related with the L_p -centroid bodies by the following proposition (see proposition 4.3 in [20]).

Proposition [20] *Let K be a symmetric convex body in \mathbb{R}^n and let $1 \leq k \leq n-1$. For every subspace F of \mathbb{R}^n of dimension k and every $q \geq 1$, we have*

$$P_F(Z_q(K)) = (k+q)^{1/q} Z_q(B_{k+q-1}) = (k+q)^{1/q} |B_{k+q-1}|^{1/k+1/q} Z_q(\widetilde{B_{k+q-1}}). \quad (8)$$

Moreover, a simple use of a result of Borell [5] gives comparison between these norms.

Lemma [5] *For f being a log-concave non-increasing function on $[0, +\infty)$, define*

$$F : t \mapsto \frac{1}{\Gamma(t)} \int_0^{+\infty} x^{t-1} f(x) dx, \quad G : t \mapsto t \int_0^{+\infty} x^{t-1} f(x) dx$$

then F is log-concave and G is log-convex on $(0, +\infty)$.

Proposition 3. *Let K be a symmetric convex body in \mathbb{R}^n , let F be a k -dimensional subspace of \mathbb{R}^n , and for any $t \geq 1$, define the symmetric convex body B_{t-1} in F as before. For every $\phi \in S_F$ and every $1 \leq s \leq t \leq u$, we have*

$$\|\phi\|_{B_{t-1}}^t \leq \frac{\Gamma(s)^{(1-\lambda)} \Gamma(u)^\lambda}{\Gamma(t)} \|\phi\|_{B_{s-1}}^{(1-\lambda)s} \|\phi\|_{B_{u-1}}^{u\lambda} \quad \text{and} \quad \|\phi\|_{B_{t-1}}^t \geq \frac{t}{s^{(1-\lambda)} u^\lambda} \|\phi\|_{B_{s-1}}^{(1-\lambda)s} \|\phi\|_{B_{u-1}}^{u\lambda}$$

where $t = (1-\lambda)s + \lambda u$.

Proof. Let $f_\phi(y) = |K \cap (E + y\phi)|$ for $y \in \mathbb{R}_+$ then by Brunn-Minkowski inequality, f_ϕ is a log-concave function and non-increasing. By Fubini, for every $\phi \in S_F$,

$$\|\phi\|_{B_{t-1}}^{-t} = \int_0^{+\infty} y^{t-1} f_\phi(y) dy = t^{-1} G(t) = \Gamma(t) F(t)$$

and the conclusion follows easily from the preceding lemma. \square

We will also use the Dvoretzky's theorem proved by Milman [17] (see also [18]).

Theorem [17] *There exist constants c_1, c_2 such that for any n , any $\varepsilon > 0$ and any symmetric convex body $L \subset \mathbb{R}^n$, if $k \leq c_1 (\varepsilon^2 / \log(1/\varepsilon)) n (W_1(L)/R(L))^2$, the set of subspaces $F \in \mathcal{G}_{n,k}$ such that*

$$(1-\varepsilon)W_1(L)D_F \subset P_F L \subset (1+\varepsilon)W_1(L)D_F$$

(where D_F is the Euclidean unit ball of F) has Haar measure greater than $1 - e^{-c_2 k}$.

Remark that it was proved by Gordon [9] that we may take ε^2 instead of $\varepsilon^2 / \log(1/\varepsilon)$.

Proof of Theorem 2. Let K be an isotropic convex body in \mathbb{R}^n . Hence from (5), for every $1 \leq q \leq c\sqrt{n}$, $R(Z_q(K)) \leq c\sqrt{q} W_1(Z_q(K))$. Without loss of generality, we can assume p to be an odd integer. Let k and $\varepsilon \in (0, 1/3)$ (to be chosen later) be such that $k^2 \leq c\varepsilon^2 n$ and $k \geq p$. Since Dvoretzky theorem holds with high probability, we can choose a subspace F of \mathbb{R}^n of dimension k such that five conditions hold simultaneously: for every $q \in \{1, p, k, 2k - p, 2k\}$,

$$(1 - \varepsilon) \frac{W_1(Z_q(K))}{W_1(Z_q(\widetilde{D}_F))} Z_q(\widetilde{D}_F) \subset P_F Z_q(K) \subset (1 + \varepsilon) \frac{W_1(Z_q(K))}{W_1(Z_q(\widetilde{D}_F))} Z_q(\widetilde{D}_F).$$

Indeed, observe that $\forall q \in \{1, p, k, 2k - p, 2k\}$, $k \leq c\varepsilon^2 n/q \leq c_1 \varepsilon^2 n (W_1(Z_q(K))/R(Z_q(K)))^2$. From (8), these inclusions mean that for every $q \in \{1, p, k, 2k - p, 2k\}$,

$$(1 - \varepsilon) \gamma_q Z_q(\widetilde{D}_F) \subset Z_q(\widetilde{B_{k+q-1}}) \subset (1 + \varepsilon) \gamma_q Z_q(\widetilde{D}_F) \quad (9)$$

where

$$\gamma_q = \frac{W_1(Z_q(K))}{(k+q)^{1/q} |B_{k+q-1}|^{1/k+1/q} W_1(Z_q(\widetilde{D}_F))}. \quad (10)$$

The first step is to prove the following

Claim: there is a universal constant c such that, for $q \in \{1, p\}$, $d(B_{k+q-1}, D_F) \leq c$.

Indeed, since B_{k+q-1} is a symmetric convex body in a k -dimensional space, it is well known that there exists a universal constant c such that $c\widetilde{B_{k+q-1}} \subset Z_q(\widetilde{B_{k+q-1}}) \subset \widetilde{B_{k+q-1}}$ for $q \geq k$ (see for example lemma 4.1 in [19] or lemma 3.1.1 in [8]). For $q \in \{k, 2k - p, 2k\}$, we deduce from (9) that $d(B_{k+q-1}, D_F) \leq c$ where c is a universal constant. Now, for $q \in \{1, p\}$, Proposition 3 with $s = k + q, t = 2k, u = 3k - q$ (i.e. $t = (1 - \lambda)s + \lambda u$ with $\lambda = 1/2$) gives

$$\begin{aligned} \|\phi\|_{B_{2k-1}}^{2k} &\leq \frac{\Gamma(k+q)^{1/2} \Gamma(3k-q)^{1/2}}{\Gamma(2k)} \|\phi\|_{B_{k+q-1}}^{(k+q)/2} \|\phi\|_{B_{3k-q-1}}^{(3k-q)/2}, \\ \|\phi\|_{B_{2k-1}}^{2k} &\geq \frac{2k}{(k+q)^{1/2} (3k-q)^{1/2}} \|\phi\|_{B_{k+q-1}}^{(k+q)/2} \|\phi\|_{B_{3k-q-1}}^{(3k-q)/2} \end{aligned}$$

for every $\phi \in S_F$. Since $q \leq p \leq k$, it is easy to conclude the proof of the claim.

The second step consists to apply Theorem 3. Indeed, for $q \in \{1, p\}$, we get from (9) that $d(Z_q(\widetilde{B_{k+q-1}}), Z_q(\widetilde{D}_F)) \leq (1 + \varepsilon)/(1 - \varepsilon) \leq 1 + 3\varepsilon$ and we have seen that $d(B_{k+q-1}, D_F) \leq c$ therefore, Theorem 3 (since q is a non even number) states that there exists a universal constant c such that

$$1 - h_k(\varepsilon) \leq \gamma_q \leq 1 + h_k(\varepsilon) \quad (11)$$

and for every $\theta, \theta_0 \in S_F$,

$$(1 + h_k(\varepsilon))^{-1} \|\theta_0\|_{B_{k+q-1}} \leq \|\theta\|_{B_{k+q-1}} \leq (1 + h_k(\varepsilon)) \|\theta_0\|_{B_{k+q-1}} \quad (12)$$

where $h_k(\varepsilon) = c^{2k} (3\varepsilon)^{1/k^2}$. We want that this last quantity goes to 0 when k goes to infinity hence we choose $\varepsilon = (2c)^{-2k^3}$ in such a way that $h_k(\varepsilon) \leq e^{-k}$. In order to use Dvoretzky theorem, k has been chosen such that $k^2 = c\varepsilon^2 n$ which means that $k \geq c'(\log n)^{1/3}$. By (10) and (11),

$$\frac{W_1(Z_p(K)) W_1(Z_1(\widetilde{D}_F))}{W_1(Z_1(K)) W_1(Z_p(\widetilde{D}_F))} \leq \frac{(1 + e^{-k})(k+p)^{1/p} |B_{k+p-1}|^{1/k+1/p}}{(1 - e^{-k})(k+1) |B_k|^{1/k}}. \quad (13)$$

We conclude observing that $|K| = 1$ can be written as

$$1 = |K| = k\omega_k \int_{S_F} \int_{K \cap E(\theta)} |\langle x, \theta \rangle|^{k-1} dx d\sigma_F(\theta) = k\omega_k \int_{S_F} \|\theta\|_{B_{k-1}}^{-k} d\sigma_F(\theta)$$

so that there exists a $\theta_0 \in S_F$ such that $1 = k\omega_k \|\theta_0\|_{B_{k-1}}^{-k}$. Using relation (12),

$$\begin{aligned} \frac{(k+p)^{1/p} |B_{k+p-1}|^{1/k+1/p}}{(k+1) |B_k|^{1+1/k}} &= \frac{(k+p)^{1/p} \left(\omega_k \int_{S_F} \|\theta\|_{B_{k+p-1}}^{-k} d\sigma_F(\theta) \right)^{1/k+1/p}}{(k+1) \left(\omega_k \int_{S_F} \|\theta\|_{B_k}^{-k} d\sigma_F(\theta) \right)^{1+1/k}} \\ &\leq \frac{(1+e^{-k})^{k+2+k/p} (k+p)^{1/p} \|\theta_0\|_{B_k}^{k+1}}{(k+1) \omega_k^{1-1/p} \|\theta_0\|_{B_{k+p-1}}^{1+k/p}}. \end{aligned}$$

Proposition 3 with $s = k, t = k+1, u = k+p$ (i.e. $t = (1-\lambda)s + \lambda u$ with $\lambda = 1/p$) gives

$$\|\theta_0\|_{B_k}^{k+1} \leq \frac{\Gamma(k)^{1-1/p} \Gamma(k+p)^{1/p}}{\Gamma(k+1)} \|\theta_0\|_{B_{k-1}}^{k(1-1/p)} \|\theta_0\|_{B_{k+p-1}}^{1+k/p}.$$

Since $\|\theta_0\|_{B_{k-1}}^k = k\omega_k$ and $p \leq k$, simple computations on the Γ function gives

$$\begin{aligned} \frac{(k+p)^{1/p} |B_{k+p-1}|^{1/k+1/p}}{(k+1) |B_k|^{1+1/k}} &\leq (1+e^{-k})^{2k} \frac{(1+p/k)^{1/p} \Gamma(k)^{1-1/p} \Gamma(k+p)^{1/p}}{(1+1/k) \Gamma(k+1)} \\ &= (1+e^{-k})^{2k} \frac{1}{k+1} \left(\frac{\Gamma(k+p+1)}{\Gamma(k+1)} \right)^{1/p} \leq 1 + cp/k. \end{aligned}$$

Combining this last inequality with (13) and with Lemma 1, we get that if $p \leq k$

$$\frac{W_1(Z_p(K)) W_1(Z_1(\tilde{D}))}{W_1(Z_1(K)) W_1(Z_p(\tilde{D}))} \leq 1 + cp/(\log n)^{1/3}$$

for a universal constant c . □

4 Stability result for L_p -centroid bodies

In Theorem 3, the equality case (i.e. $\varepsilon = 0$) may be treated via the use of Funk-Hecke theorem. This is why we will follow an approach using the decomposition in spherical harmonics and we refer to the chapter 3 of the book of Groemer [10] for more detailed explanation. This technique was also used by Bourgain and Lindenstrauss [6].

Let p be an odd integer with $p \leq d$, we consider the function $\phi : \mathbb{R} \rightarrow \mathbb{R}$ defined by $\phi(t) = |t|^p$ and we define the operator J_ϕ on $L_2(S^{d-1})$ by

$$J_\phi(F)(u) = \int_{S^{d-1}} \phi(\langle u, v \rangle) F(v) d\sigma(v)$$

for any $u \in S^{d-1}$. By the Funk-Hecke theorem, for every harmonic polynomial H homogeneous of degree l on the sphere S^{d-1} we have $\langle J_\phi(F), H \rangle = \alpha_{d,l}(\phi) \langle F, H \rangle$, where $\langle \cdot, \cdot \rangle$ denotes the usual scalar product in $L_2(S^{d-1})$ and

$$\alpha_{d,l}(\phi) = \frac{(-1)^l \pi^{(d-1)/2}}{2^{l-1} \Gamma(l + \frac{d-1}{2})} \int_{-1}^1 \phi(t) \frac{d^l}{dt^l} (1-t^2)^{(l+\frac{d-3}{2})} dt.$$

These coefficients are known, see [21] or Lemma 1 in [13]. Hence, for any odd values of l , $\alpha_{d,l}(\phi) = 0$ and for any even values of l ,

$$\alpha_{d,l}(\phi) = \frac{\pi^{d/2-1} \Gamma(p+1) \sin(\pi(l-p)/2) \Gamma((l-p)/2)}{2^{p-1} \Gamma((l+d+p)/2)}.$$

Standard computations with the Γ function gives a universal constant c such that for any even integer l ,

$$\frac{1}{\alpha_{d,l}(\phi)^{1/(p+d/2)}} \leq c \max(d, l). \quad (14)$$

For a continuous function $F : S^{d-1} \rightarrow \mathbb{R}$ such that $F^\vee : \mathbb{R}^d \rightarrow \mathbb{R}$ defined by $F^\vee(x) = F(x/|x|_2)$ is differentiable on $\mathbb{R}^d \setminus \{0\}$, we set for any $u \in S^{d-1}$, $\nabla_0 F(u) = \nabla F^\vee(u)$. The next proposition is a standard trick using spherical harmonics [10].

Proposition 4. *There exists a universal constant c such that for any continuous even function $F : S^{d-1} \rightarrow \mathbb{R}$ such that $\nabla_0 F$ exists,*

$$\|F\|_2 \leq c \|J_\phi(F)\|_2^{2/(d+2p+2)} \left(\|\nabla_0 F\|_2^2 + d^2 \|F\|_2^2 \right)^{\frac{1}{2}(1-2/(d+2p+2))}.$$

Proof. Let $F \sim \sum Q_l(F)$ be the decomposition in spherical harmonics of F (with $Q_l(F)$ spherical harmonics of degree l) then by Corollary 3.2.12 in [10]

$$\|\nabla_0 F\|_2^2 = \sum_{l \geq 0} l(l+d-2) \|Q_l(F)\|_2^2.$$

For any odd l , $\alpha_{d,l}(\phi) = 0$ and since F is even, $Q_l(F) = 0$. Hence from Parseval equality

$$\|F\|_2^2 = \sum_{l \text{ even}} \|Q_l(F)\|_2^2 = \sum_{l \text{ even}} (\alpha_{d,l}(\phi) \|Q_l(F)\|_2)^\beta \|Q_l(F)\|_2^{2-\beta} \alpha_{d,l}(\phi)^{-\beta}$$

where $\beta \in (0, 2)$ is chosen such that $2\beta/(2-\beta) = 2/(p+d/2)$. By Hölder inequality,

$$\|F\|_2^2 \leq \left(\sum_{l \text{ even}} \alpha_{d,l}(\phi)^2 \|Q_l(F)\|_2^2 \right)^{\beta/2} \left(\sum_{l \text{ even}} \|Q_l(F)\|_2^2 \alpha_{d,l}(\phi)^{-2\beta/(2-\beta)} \right)^{1-\beta/2}.$$

By Funk-Hecke theorem, $\|J_\phi(F)\|_2^2 = \sum_{l \text{ even}} \alpha_{d,l}(\phi)^2 \|Q_l(F)\|_2^2$ and by inequality (14),

$$\begin{aligned} \sum_{l \text{ even}} \|Q_l(F)\|_2^2 \alpha_{d,l}(\phi)^{-2/(p+d/2)} &\leq c^2 \sum_{l \text{ even}} \max(d^2, l^2) \|Q_l(F)\|_2^2 \\ &\leq c^2 \left(d^2 \sum_{l \text{ even}, l \leq d} \|Q_l(F)\|_2^2 + \sum_{l \text{ even}, l \geq d} l^2 \|Q_l(F)\|_2^2 \right) \leq c^2 \left(d^2 \|F\|_2^2 + \|\nabla_0(F)\|_2^2 \right). \end{aligned}$$

This proves that $\|F\|_2 \leq c \|J_\phi(F)\|_2^{2/(d+2p+2)} \left(\|\nabla_0 F\|_2^2 + d^2 \|F\|_2^2 \right)^{\frac{1}{2}(1-2/(d+2p+2))}$. \square

We will also need the following simple lemma.

Lemma 5. *Let $F : S^{d-1} \rightarrow \mathbb{R}$ be a Lipschitz function and let $M = \max(\|F\|_2, \|F\|_{\text{Lip}})$ then*

$$\|F\|_\infty \leq 5M^{(d-1)/(d+1)} \|F\|_2^{2/(d+1)}.$$

Proof. Let $u \in S^{d-1}$ such that $|F(u)| = \|F\|_\infty$ and let $C(u, R)$ be the spherical cap of radius R centered at u . For any $\delta \geq 1$, define $A_\delta = \{v \in S^{d-1}, |F(v)| \leq \delta \|F\|_2\}$ then by Chebychev inequality, $\sigma(A_\delta) \geq 1 - 1/\delta^2$. For any $R \in (0, 2)$, it is well known that $\sigma(C(u, R)) \geq \frac{1}{2} \left(\frac{R}{2}\right)^{d-1}$. If R is chosen such that $\frac{1}{2} \left(\frac{R}{2}\right)^{d-1} = \frac{1}{\delta^2}$ then $A_\delta \cap C(u, R) \neq \emptyset$. In that case, take $v \in A_\delta \cap C(u, R)$ then

$$|F(u)| \leq |F(u) - F(v)| + |F(v)| \leq \|F\|_{\text{Lip}} |u - v|_2 + \delta \|F\|_2 \leq RM + \delta \|F\|_2.$$

Since $R = 2(2/\delta^2)^{1/(d-1)}$, we get the estimate taking $\delta = (M/\|F\|_2)^{(d-1)/(d+1)} \geq 1$. \square

Proof of Theorem 3. Using support functions, $d(Z_p(\tilde{K}), Z_p(\tilde{D})) \leq 1 + \varepsilon$ implies that there exists $\gamma > 0$ such that

$$\gamma h_{Z_p(\tilde{D})} \leq h_{Z_p(\tilde{K})} \leq (1 + \varepsilon)\gamma h_{Z_p(\tilde{D})}. \quad (15)$$

For any symmetric convex body $L \subset \mathbb{R}^d$, by integration in polar coordinates,

$$h_{Z_p(L)}(u)^p = \int_L |\langle x, u \rangle|^p dx = \frac{d\omega_d}{d+p} \int_{S^{d-1}} |\langle v, u \rangle|^p \frac{1}{\|v\|_L^{d+p}} d\sigma(v)$$

hence applying it for $L = \tilde{K}$ and $L = \tilde{D}$, we get for any $u \in S^{d-1}$,

$$\left| \int_{S^{d-1}} |\langle v, u \rangle|^p \left(\frac{1}{\|v\|^{d+p}} - \frac{\gamma^p}{\omega_d^{1+p/d}} \right) d\sigma(v) \right| \leq ((1 + \varepsilon)^p - 1) \frac{\gamma^p}{\omega_d^{1+p/d}} \int_{S^{d-1}} |\langle v, u \rangle|^p d\sigma(v)$$

where $\|\cdot\|$ is the norm with unit ball \tilde{K} . For every $u \in S^{d-1}$, let $F(u) = \frac{\omega_d^{1+p/d}}{\gamma^p \|u\|^{p+d}} - 1$. Since $\forall u \in S^{d-1}$, $\int_{S^{d-1}} |\langle v, u \rangle|^p d\sigma(v) \leq 1$, we get

$$\|J_\phi(F)\|_2 \leq \|J_\phi(F)\|_\infty \leq ((1 + \varepsilon)^p - 1). \quad (16)$$

Since $d(K, D) \leq \alpha$, there exists $a, b > 1$ such that $1/a\tilde{D} \subset \tilde{K} \subset b\tilde{D}$ and $ab = \alpha$.

$$\forall y \in S^{d-1}, \gamma^p (1 + \varepsilon)^p h_{Z_p(\tilde{D})}^p(y) \geq h_{Z_p(\tilde{K})}^p(y) \geq \int_{\tilde{D}/a} |\langle x, y \rangle|^p dx = h_{Z_p(\tilde{D})}^p(y)/a^{d+p}$$

therefore $1/\gamma^p \leq a^{d+p}(1 + \varepsilon)^p$. For any $x \in \mathbb{R}^d$, $\omega_d^{1/d} b^{-1} |x|_2 \leq \|x\| \leq a\omega_d^{1/d} |x|_2$ and for $u \in S^{d-1}$, $\nabla F^\vee(u) = \frac{\omega_d^{1+p/d}}{\gamma^p} \frac{(p+d)}{\|u\|^{p+d}} \left(u - \frac{\nabla \|\cdot\|(u)}{\|u\|} \right)$, therefore

$$\|\nabla_0 F\|_2 \leq \|\nabla_0 F\|_\infty \leq \frac{(p+d)b^{p+d}}{\gamma^p} (1 + ab) \leq 4 d \alpha^{d+p+1} (1 + \varepsilon)^p. \quad (17)$$

We also have $\|F\|_2 \leq \|F\|_\infty \leq 1 + b^{p+d}/\gamma^p \leq 2\alpha^{p+d}(1 + \varepsilon)^p$. Using Proposition 4 with (16) and (17), we get

$$\|F\|_2 \leq c((1 + \varepsilon)^p - 1)^{2/(d+2p+2)} (6 d \alpha^{d+p+1} (1 + \varepsilon)^p)^{1-2/(d+2p+2)} \leq c\varepsilon^{2/(d+2p+2)} (4\alpha)^{d+p+1}.$$

Moreover, for any $u, v \in S^{d-1}$, $F(u) - F(v) = \omega_d^{1+p/d}/\gamma^p (1/\|u\|^{p+d} - 1/\|v\|^{p+d})$ and

$$|F(u) - F(v)| \leq \frac{\omega_d^{1+p/d}}{\gamma^p} \|u - v\| \sum_{i=0}^{d+p-1} \|u\|^{-(d+p-i)} \|v\|^{-(i+1)} \leq 2 d \alpha^{d+p+1} (1 + \varepsilon)^p \|u - v\|_2.$$

Therefore $\max(\|F\|_2, \|F\|_{\text{Lip}}) \leq (4\alpha)^{d+p+1}$ and by Lemma 5,

$$\|F\|_\infty \leq c(4\alpha)^{d+p+1} \varepsilon^{4/(d+1)(d+2p+2)} \leq c(4\alpha)^{d+p+1} \varepsilon^{1/d^2} := f(\varepsilon).$$

Recalling the definition of F , $F(u) = -1 + \omega_d^{1+p/d}/\gamma^p \|u\|^{p+d}$, $\forall u \in S^{d-1}$, we have proved

$$(1 - f(\varepsilon))^{1/d+p} \gamma^{p/d+p} \tilde{D} \subset \tilde{K} \subset (1 + f(\varepsilon))^{1/d+p} \gamma^{p/d+p} \tilde{D}. \quad (18)$$

Since $|\tilde{K}| = |\tilde{D}| = 1$, $(1 + f(\varepsilon))^{-1} \leq \gamma^p \leq (1 - f(\varepsilon))^{-1}$ and choosing $\varepsilon \leq (c\alpha)^{-2d^3}$, (15) and (18) proves the assertions of Theorem 3. \square

References

- [1] M. Anttila, K. Ball and I. Perissinaki, *The central limit problem for convex bodies*, Trans. Amer. Math. Soc. **355** (2003), no. 12, 4723–4735.
- [2] K. Ball, *Logarithmically concave functions and sections of convex sets in R^n* , Studia Math. **88** (1988), no. 1, 69–84.
- [3] S. G. Bobkov, *Remarks on the growth of L^p -norms of polynomials*, in *Geometric Aspects of Functionnal Analysis*, pp. 27–35, Lecture Notes in Math. 1745, Springer, Berlin, 2000.
- [4] S. G. Bobkov, *On isoperimetric constants for log-concave probability distributions*, to appear in *Geometric Aspects of Functionnal Analysis*, Springer, Berlin, 2006.
- [5] C. Borell, *Complements of Lyapunov’s inequality*, Math. Ann. **205** (1973), 323–331.
- [6] J. Bourgain and J. Lindenstrauss, *Projection bodies*, in *Geometric aspects of functional analysis (1986/87)*, 250–270, Lecture Notes in Math., 1317, Springer, Berlin, 1988.
- [7] U. Brehm and J. Voigt, *Asymptotics of cross sections for convex bodies*, Beiträge Algebra Geom. **41** (2000), no. 2, 437–454.
- [8] A. Giannopoulos, *Notes on isotropic convex bodies*, Institute of Mathematics, Polish Academy of Sciences, Warsaw (2003), <http://users.uoa.gr/apgiannop/isotropic-bodies.ps>
- [9] Y. Gordon, *Gaussian processes and almost spherical sections of convex bodies*, Ann. Probab. **16** (1988), no. 1, 180–188.
- [10] H. Groemer, *Geometric applications of Fourier series and spherical harmonics*, Cambridge Univ. Press, Cambridge, 1996.
- [11] R. Kannan, L. Lovász and M. Simonovits, *Isoperimetric problems for convex bodies and a localization lemma*, Discrete Comput. Geom. **13** (1995), no. 3-4, 541–559.
- [12] B. Klartag, *A central limit theorem for convex sets*, to appear in Invent. Math.
- [13] A. Koldobsky, *Common subspaces of L_p -spaces*, Proc. Amer. Math. Soc. **122** (1994), 207–212.
- [14] A. E. Litvak, V. D. Milman and G. Schechtman, *Averages of norms and quasi-norms*, Math. Ann. **312** (1998), no. 1, 95–124.
- [15] E. Lutwak, D. Yang, G. Zhang, *L_p affine isoperimetric inequalities*, J. Differential Geom. **56** (2000), no. 1, 111–132.
- [16] E. Lutwak, G. Zhang, *Blaschke-Santaló inequalities*, J. Differential Geom. **47** (1997), no. 1, 1–16.
- [17] V. D. Milman, *A new proof of A. Dvoretzky’s theorem on cross-sections of convex bodies*, (Russian), Funkcional. Anal. i Priložen. **5** (1971), no. 4, 28–37.
- [18] V. D. Milman and G. Schechtman, *Asymptotic theory of finite-dimensional normed spaces*, Lecture Notes in Mathematics, **1200**, Springer, Berlin, 1986.
- [19] G. Paouris, *Ψ_2 -estimates for linear functionals on zonoids*, in *Geometric Aspects of Functional Analysis*, 211–222, Lecture Notes in Math., 1807, Springer, Berlin, 2003.
- [20] G. Paouris, *Concentration of mass in convex bodies*, Geom. Funct. Anal., **16** (2006), no.5, 1021–1049.
- [21] D. St. P. Richards, *Positive definite symmetric functions on finite dimensional spaces. 1. Applications of the Radon transform*, J. Multivariate Anal **19** (1986), 280–298.