Math 411 - Fall 2014 - Practice for the final Exam.

1. Suppose that the number of units produced daily at factory A is a random variable with mean 2 and standard deviation 1 and the number produced at factory B is a random variable with mean 3 and standard deviation 1.

   a. Estimate (from above) the probability that more than 7 units are produced today.

   b. Assume independence and estimate again the above probability using Chebychev’s inequality.

2. To complete a job two separate steps are required, with the time needed for the first step being an exponential random variable with expectation 1 and the time for the second step being an independent exponential random variable with expectation 2. Use the central limit theorem to approximate the probability that 20 jobs will be completed in less than 75 hours.

3. Pick a point $X$ uniformly on $[0,1]$ and then pick a point $Y$ uniformly on $[0,X]$.

   a. Compute the densities $f_X$ and $f_{Y|X}(y|x)$.

   b. Find the joint density of $(X,Y)$.

   c. Compute the densities $f_Y$ and $f_{X|Y}(x|y)$.

   d. Is it true that $X$ is uniform on $[Y,1]$? Justify your answer.

4. Let $X$ and $Y$ be continuous random variables with joint density function

   \[(*) \quad f(x, y) = ce^{-y}, \quad 0 < x < y < \infty.\]

   a. Find $c$.

   b. Compute the marginal densities.

   c. Is it true that $X, Y$ are independent?

   d. Compute the conditional densities $f_{X|Y}(x|y), f_{Y|X}(y|x)$.

   e. Compute the probability $P(0 \leq X \leq 1|Y = 1)$.

   f. Let $Z_1, Z_2$ two independent exponentials with parameter 1 each. Let $X = Z_1$ and $Y = Z_1 + Z_2$. Show that $(X,Y)$ have joint density as in $(*)$.

5. The joint probability mass function of the random variables $X,Y$ is

\[
p(0,-1) = p(0,0) = p(0,1) = p(1,0) = \frac{1}{4}, \quad p(1,-1) = p(1,1) = 0.
\]
a. Compute the marginal probability mass functions of X and Y.
b. Is it true that X and Y are independent?
c. Compute $E(XY), E(X), E(Y)$.
d. Compute the $Cov(X, Y)$.

6. Let X, Y, Z be independent and uniformly distributed on (0, 1).
   a. Compute the probability $P(X \leq \frac{Y+Z}{2})$.
   b. Compute the expectation of the random variable $Z = e^{XY^2}Z$.

7. Let X be a non-negative random variable with expectation 1.
   a. Estimate the probability $P(X \geq 4)$ from above.
   b. Assume that the variance of X is 1. Use this information to estimate again the above probability.
   c. Assume that $E(X^3) = 3$. Use this information to estimate again the probability in [a].
   d. Assume that X is exponential with parameter 1. Compute the probability in [a] and compare your answer with the estimates in [a], [b].

8. The number of years that a machine functions is a random variable whose hazard rate function is given by

   \[ \lambda(t) := \begin{cases} 
   0.2 & 0 < t \leq 2 \\
   0.2 + 0.3(t - 2) & 2 < t \leq 5 \\
   1.1 & t > 5 
\end{cases} \]

   1. What is the probability that the machine will still be working 6 years after being purchased?
   2. If it is still working 6 years after purchased, what is the conditional probability that it will fail within the next 2 years?

9. Let $X_1, X_2, \cdots$ be non-negative, independent and identically distributed random variables. Assume that $E(\log X_i) < \infty$. Use the law of large numbers to show that

   \[ \lim_{n \to \infty} \left( \prod_{i=1}^{n} X_i \right)^{\frac{1}{n}} = e^{E(\log X_i)} \]

   with probability 1.
Hints/solutions

1. 1.a Note that $E[X] = 5$. By Markov’s inequality $P(X \geq 7) \leq \frac{5}{7}$.
   
   1.b. Note that $\text{var}(X) = 2$. Then by Chebychev’s inequality
   
   $$P(X \geq 7) \leq P(|X - 5| \geq 2) \leq \frac{1}{2}.$$ 

2. Let $W$ be the r.v. of the time required to finish the job. Then $E[W] = 3$ and $\text{var}(W) = 3$. So
   
   $$P(W_1 + \cdots + W_2 \geq 75) = P\left(\frac{W_1 + \cdots + W_2 - 60}{\sqrt{60}} \geq \frac{15}{\sqrt{60}}\right) \simeq \text{CLT} 1 - \Phi\left(\frac{15}{\sqrt{60}}\right).$$

3. a. By definition $f_{Y|X}(y|x) = \frac{1}{x}1_{[0,x]}(y)$ and $f_X(x) := 1_{[0,1]}(x)$.
   
   b. $f_{X,Y}(x,y) = \frac{1}{x}$, if $0 \leq y \leq x \leq 1$ and 0 otherwise.
   
   c. $f_Y(y) := \int_y^1 \frac{1}{x}dx = \log \frac{1}{y}$.
   
   d. No, since $f_{X|Y}(x|y) = (x \log \frac{1}{y})^{-1}$ if $0 \leq y \leq x \leq 1$.

4. a. $c = 1$. b. $f_X(x) = \int_x^\infty e^{-y}dy = e^{-x}$. $f_Y(y) = ye^{-y}$. c. No
   
   d. $f_{X|Y}(x|y) = \frac{1}{y}$ if $0 \leq x \leq y$. e. Probability is equal to 1.
   
   f. (Hint) Show that the joint moment generating function of $(X,Y)$ (at $t_1,t_2$) is equal to $\frac{1}{(1-t_1-t_2)(1-t_2)}$.

5. $p_X(0) = \frac{3}{4}, p_X(1) = \frac{1}{4}, p_Y(-1) = p_Y(1) = \frac{1}{4}, p_Y(0) = \frac{2}{4}$.
   
   b. No since $\frac{1}{4} = p_{X,Y}(0,1) \neq p_X(0)p_Y(1)$. c. $E[XY] = E[X] = E[Y] = 0$.
   
   d. Cov$(X,Y) = 0$.

6. $P(X \leq \frac{Y+Z}{2}) = \int_0^1 \int_0^1 \int_0^{\frac{y+z}{2}} dxdydz$.
   
   b. $E[e^{X Y^2 Z}] = E[e^X][E[Y^2]][E[Z]]$.

7. a. By Markov’s inequality $P(X \geq 4) \leq \frac{1}{4}$.
   
   b. By Chebychev’s inequality $P(X \geq 4) \leq P(|X - 1| \geq 3) \leq \frac{1}{9}$.
   
   c. Let $W := X^3$. Then by Markov’s inequality for $t > 0$,
   
   $$P(X \geq t 3^{\frac{1}{2}}) = P(W \geq 3t^3) \leq \frac{1}{t^3}.$$ 

   Apply for $t = \frac{4}{3\pi}$. 

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8. Use formula (5.4) in page 213.

9. Set $L_i := \log X_i$ and note that $L_i$ are i.i.d. random variables. Apply the Law of large numbers for $L_i$'s.