

# $\Psi_\alpha$ -ESTIMATES FOR MARGINALS OF LOG-CONCAVE PROBABILITY MEASURES

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ABSTRACT. We show that a random marginal  $\pi_F(\mu)$  of an isotropic log-concave probability measure  $\mu$  on  $\mathbb{R}^n$  exhibits better  $\psi_\alpha$ -behavior:

- (i) If  $k \leq \sqrt{n}$ , then for a random  $F \in G_{n,k}$  we have that  $\pi_F(\mu)$  is a  $\psi_2$ -measure. We complement this result by showing that a random  $\pi_F(\mu)$  is, at the same time, supergaussian.
- (ii) If  $k = n^\delta$ ,  $\frac{1}{2} < \delta < 1$ , then for a random  $F \in G_{n,k}$  we have that  $\pi_F(\mu)$  is a  $\psi_{\alpha(\delta)}$ -measure, where  $\alpha(\delta) = \frac{2\delta}{3\delta-1}$ .

## 1. INTRODUCTION

The purpose of this note is to provide estimates on the  $\psi_\alpha$ -behavior of random marginals of log-concave probability measures. We show that random  $k$ -dimensional projections of a high-dimensional measure of the log-concave class have better tail properties than the original measure. We give precise quantitative estimates for every  $1 \leq k < n$ . A typical  $k$ -dimensional marginal is  $\psi_2$  as long as  $k \leq \sqrt{n}$ ; after this critical value we still have non-trivial information ( $\alpha$  is always greater than a simple function of  $\frac{\log n}{\log k}$ ) in full generality. This observation may be viewed as a continuation of the ideas and the tools that were developed in [17]. It is also parallel to the philosophy behind Klartag's proof of the central limit theorem for convex bodies in [7] and [8] (see also [5] and [4]). A main ingredient in these works is the fact that appropriate marginals of log-concave measures in power-type dimensions ( $k \simeq n^\epsilon$  for some  $\epsilon > 0$ ) are approximately spherically-symmetric. As Klartag proves in [9] this phenomenon appears for a much wider class of probability measures and constitutes the measure analogue of Dvoretzky's theorem on approximately Euclidean sections of high-dimensional convex bodies. Actually, Dvoretzky's theorem plays a crucial role in all these works, as well as in the present note.

Recall that a probability measure  $\mu$  on  $\mathbb{R}^n$  is called log-concave if for any Borel sets  $A, B$  in  $\mathbb{R}^n$  and any  $\lambda \in (0, 1)$ ,

$$(1.1) \quad \mu(\lambda A + (1 - \lambda)B) \geq \mu(A)^\lambda \mu(B)^{1-\lambda}.$$

It is known (see [2]) that if  $\mu$  is log-concave and if  $\mu(H) < 1$  for every hyperplane  $H$ , then  $\mu$  has a density  $f = f_\mu$ , with respect to the Lebesgue measure, which is log-concave:  $\log f$  is concave on its support  $\{f > 0\}$ .

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We say that  $\mu$  is isotropic if it is centered, i.e.

$$(1.2) \quad \int_{\mathbb{R}^n} \langle x, \theta \rangle f(x) dx = 0,$$

and satisfies the isotropic condition

$$(1.3) \quad \int_{\mathbb{R}^n} \langle x, \theta \rangle^2 f(x) dx = 1$$

for all  $\theta \in S^{n-1}$ . Then, the isotropic constant of  $\mu$  is defined by  $L_\mu := f(0)^{1/n}$ .

Let  $1 \leq \alpha \leq 2$ . We say that a direction  $\theta \in S^{n-1}$  is a  $\psi_\alpha$ -direction for  $\mu$  with constant  $r > 0$  if

$$(1.4) \quad \|\langle \cdot, \theta \rangle\|_{\psi_\alpha} \leq r \|\langle \cdot, \theta \rangle\|_2,$$

where

$$(1.5) \quad \|u\|_{\psi_\alpha} = \inf \left\{ t > 0 : \int_{\mathbb{R}^n} \exp((|u(x)|/t)^\alpha) f(x) dx \leq 2 \right\}.$$

We say that  $\mu$  is a  $\psi_\alpha$  measure with constant  $r > 0$  if (1.4) holds true for every  $\theta \in S^{n-1}$ . It is well known that there exists an absolute constant  $C > 0$  such that every log-concave probability measure  $\mu$  is  $\psi_1$  with constant  $C$ .

We study the  $\psi_\alpha$ -behavior of marginals of  $\mu$ . For every integer  $1 \leq k < n$  and any  $F \in G_{n,k}$ , we consider the measure  $\pi_F(\mu)$  with density

$$(1.6) \quad \pi_F(f)(x) = \int_{x+F^\perp} f(y) dy.$$

By the Prékopa–Leindler inequality (see [20]),  $\pi_F(\mu)$  is a log-concave probability measure on  $F$ . As a simple consequence of Fubini's theorem, one can check that if  $\mu$  is isotropic then  $\pi_F(\mu)$  is also isotropic.

For the study of marginals, we need a variant of the  $\psi_\alpha$  norm. We start with the well-known fact that  $\|u\|_{\psi_\alpha} \simeq \sup \left\{ \frac{\|u\|_q}{q^{1/\alpha}} : q \geq \alpha \right\}$  and recall that if  $\mu$  is the Lebesgue measure  $\mu_K$  on an isotropic convex body  $K$  in  $\mathbb{R}^n$  and if  $u$  is a linear functional, then

$$(1.7) \quad \|u\|_{\psi_\alpha} \simeq \sup_{q \geq \alpha} \frac{\|u\|_q}{q^{1/\alpha}} \simeq \sup_{\alpha \leq q \leq n} \frac{\|u\|_q}{q^{1/\alpha}}.$$

We define

$$(1.8) \quad \|u\|_{\psi'_\alpha} = \sup_{\alpha \leq q \leq n} \frac{\|u\|_q}{q^{1/\alpha}}.$$

It is clear that  $\|u\|_{\psi'_\alpha} \ll \|u\|_{\psi_\alpha}$ . In view of (1.7) this is a natural definition of a “ $\psi_\alpha$ -norm” when one studies the behavior of linear functionals with respect to a log-concave measure on  $\mathbb{R}^n$ ; see, for example, the applications in Section 4.

Our first result provides estimates on the  $\psi'_\alpha$ -behavior of random marginals of  $\mu$ .

**Theorem 1.1.** *Let  $\mu$  be an isotropic log-concave probability measure on  $\mathbb{R}^n$ .*

- (i) *If  $k \leq \sqrt{n}$  then there exists  $A_k \subseteq G_{n,k}$  with measure  $\nu_{n,k}(A_k) > 1 - \exp(-c\sqrt{n})$  such that, for every  $F \in A_k$ ,  $\pi_F(\mu)$  is a  $\psi'_2$ -measure with constant  $C$ , where  $C > 0$  is an absolute constant.*

- (ii) If  $k = n^\delta$ ,  $\frac{1}{2} < \delta < 1$  then there exists  $A_k \subseteq G_{n,k}$  with measure  $\nu_{n,k}(A_k) > 1 - \exp(-ck)$  such that, for every  $F \in A_k$ ,  $\pi_F(\mu)$  is a  $\psi'_{\alpha(\delta)}$ -measure with constant  $C$ , where  $\alpha(\delta) = \frac{2\delta}{3\delta-1}$  and  $C > 0$  is an absolute constant.

We next consider the question whether, in the case  $1 \leq k \leq \sqrt{n}$ , random marginals  $\pi_F(\mu)$  of an isotropic log-concave probability measure  $\mu$  on  $\mathbb{R}^n$  are supergaussian (in the terminology of [19]). If  $\nu$  is an isotropic log-concave probability measure on  $\mathbb{R}^k$ , a direction  $\theta \in S^{k-1}$  is called supergaussian for  $\nu$  with constant  $r > 0$  if, for all  $1 \leq t \leq \frac{\sqrt{k}}{r}$ ,

$$(1.9) \quad \nu(\{x : |\langle x, \theta \rangle| \geq t\}) \geq e^{-r^2 t^2}.$$

The minimum of the set of  $r > 0$  for which (1.9) holds true is called the supergaussian constant of  $\nu$  in the direction of  $\theta$  and is denoted by  $\overline{sg}_\nu(\theta)$ . It was proved in [19] that if  $K$  is an isotropic convex body in  $\mathbb{R}^k$ , then a random direction is supergaussian for  $\nu_K$  with a constant  $O(L_K)$  (the same question had been considered by Pivovarov [21] for the class of 1-unconditional bodies). We prove the following.

**Theorem 1.2.** *Let  $\mu$  be an isotropic log-concave probability measure on  $\mathbb{R}^n$ . If  $k \leq \sqrt{n}$ , then there exists  $B_k \subseteq G_{n,k}$  with measure  $\nu_{n,k}(B_k) > 1 - \exp(-c\sqrt{n})$  such that, for every  $F \in B_k$ ,  $\pi_F(\mu)$  is a supergaussian measure with constant  $c$ , where  $c > 0$  is an absolute constant: this means that*

$$(1.10) \quad \inf_{\theta \in S_F} \overline{sg}_{\pi_F(\mu)}(\theta) \geq c.$$

The paper is organized as follows. In Section 2 we introduce background material on  $L_q$ -centroid bodies; these play a central role in our approach. The proof of the two main results is presented in Section 3. Generalizations, applications and further remarks are collected in Section 4.

*Notation and Preliminaries.* We work in  $\mathbb{R}^n$ , which is equipped with a Euclidean structure  $\langle \cdot, \cdot \rangle$ . We denote by  $\|\cdot\|_2$  the corresponding Euclidean norm, and write  $B_2^n$  for the Euclidean unit ball, and  $S^{n-1}$  for the unit sphere. Volume is denoted by  $|\cdot|$ . We write  $\sigma$  for the rotationally invariant probability measure on  $S^{n-1}$ . The Grassmann manifold  $G_{n,k}$  of  $k$ -dimensional subspaces of  $\mathbb{R}^n$  is equipped with the Haar probability measure  $\nu_{n,k}$ . We also write  $\tilde{A}$  for the homothetic image of volume 1 of a compact set  $A \subseteq \mathbb{R}^n$ , i.e.  $\tilde{A} := \frac{A}{|A|^{1/n}}$ .

The letters  $c, c', c_1, c_2$  etc. denote absolute positive constants which may change from line to line. Whenever we write  $a \simeq b$ , we mean that there exist absolute constants  $c_1, c_2 > 0$  such that  $c_1 a \leq b \leq c_2 a$ . We refer to [14], [6] and [18] for information on isotropic convex bodies and to the books [15] and [20] for the asymptotic theory of finite dimensional normed spaces.

A convex body in  $\mathbb{R}^n$  is a compact convex subset  $C$  of  $\mathbb{R}^n$  with non-empty interior. We say that  $C$  is symmetric if  $x \in C$  implies that  $-x \in C$ . We say that  $C$  is centered if  $\int_C \langle x, \theta \rangle dx = 0$  for every  $\theta \in S^{n-1}$ . The support function  $h_C : \mathbb{R}^n \rightarrow \mathbb{R}$  of  $C$  is defined by  $h_C(x) = \max\{\langle x, y \rangle : y \in C\}$ . For each  $-\infty < p < \infty$ ,  $p \neq 0$ , we define the  $p$ -mean width of  $C$  by

$$(1.11) \quad w_p(C) = \left( \int_{S^{n-1}} h_C^p(\theta) \sigma(d\theta) \right)^{1/p}.$$

Note that  $w(C) := w_1(C)$  is the mean width of  $C$ . The radius of  $C$  is the quantity  $R(C) = \max\{\|x\|_2 : x \in C\}$  and, if the origin is an interior point of  $C$ , the polar

body of  $C$  is  $C^\circ := \{y \in \mathbb{R}^n : \langle x, y \rangle \leq 1 \text{ for all } x \in C\}$ . If  $K$  is a convex body in  $\mathbb{R}^n$  then the Brunn-Minkowski inequality implies that the measure  $\mu_K$  with density  $\mathbf{1}_{\tilde{K}}$  is log-concave.

## 2. BASIC FORMULAS

**2.1.** Let  $\mu$  be a log-concave probability measure on  $\mathbb{R}^n$  with a log-concave density  $f$ . For every  $q \geq 1$  and  $y \in \mathbb{R}^n$  we define

$$(2.1) \quad h_{Z_q(\mu)}(y) := \left( \int_{\mathbb{R}^n} |\langle x, y \rangle|^q f(x) dx \right)^{1/q}.$$

The integral is finite for every  $q \geq 1$ , by the log-concavity of  $\mu$ . We define the  $L_q$ -centroid body  $Z_q(\mu)$  of  $\mu$  to be the centrally symmetric convex set with support function  $h_{Z_q(\mu)}$ .

$L_q$ -centroid bodies were introduced in [11]. The normalization and notation was different (see also [12] where an  $L_q$  affine isoperimetric inequality was proved). We follow the normalization and notation of [17]. If  $K$  is a convex body of volume 1, we also write  $Z_q(K)$  instead of  $Z_q(\mu_K)$ .

It is a simple consequence of Hölder's inequality that  $Z_p(\mu) \subseteq Z_q(\mu)$  for all  $1 \leq p \leq q < \infty$ . On the other hand, Borell's lemma (see [15]) implies that

$$(2.2) \quad Z_q(\mu) \subseteq \bar{c}_0 \frac{q}{p} Z_p(\mu)$$

for all  $1 \leq p < q < \infty$ , where  $\bar{c}_0 \geq 1$  is an absolute constant. For additional information on  $L_q$ -centroid bodies, we refer to [17] and [18].

**2.2.** Let  $\mu$  be a log-concave probability measure on  $\mathbb{R}^n$  with a log-concave density  $f$ , and let  $1 \leq k \leq n$  and  $F \in G_{n,k}$ . Fubini's theorem shows that, for every  $q \geq 1$  and  $\theta \in S_F$ ,

$$(2.3) \quad \int_{\mathbb{R}^n} |\langle x, \theta \rangle|^q d\mu(x) = \int_F |\langle x, \theta \rangle|^q d\pi_F(\mu)(x).$$

Since  $h_{P_F(Z_q(\mu))}(\theta) = h_{Z_q(\mu)}(\theta)$  for all  $\theta \in S_F$ , it follows that

$$(2.4) \quad P_F(Z_q(\mu)) = Z_q(\pi_F(\mu)).$$

**2.3.** Let  $\mu$  be a log-concave centered probability measure on  $\mathbb{R}^n$ . For every  $q > -n$ ,  $q \neq 0$ , we define the quantities  $I_q(\mu)$  by

$$(2.5) \quad I_q(\mu) := \left( \int_{\mathbb{R}^n} \|x\|_2^q d\mu(x) \right)^{1/q}.$$

The following fact is proved in [18]: For every  $1 \leq q \leq n/2$ ,

$$(2.6) \quad I_{-q}(\mu) \simeq \sqrt{n/q} w_{-q}(Z_q(\mu))$$

and

$$(2.7) \quad I_q(\mu) \simeq \sqrt{n/q} w_q(Z_q(\mu)).$$

**2.4.** Let  $C$  be a symmetric convex body in  $\mathbb{R}^n$ . Define  $k_*(C)$  as the largest positive integer  $k \leq n$  for which a random  $k$ -dimensional projection of  $C$  is 4-Euclidean:

this can be made precise if we ask, for example, that the measure of the set of  $F \in G_{n,k}$  which satisfy

$$(2.8) \quad \frac{1}{2}W(C)(B_2^n \cap F) \subseteq P_F(C) \subseteq 2W(C)(B_2^n \cap F)$$

is greater than  $\frac{n}{n+k}$ . The parameter  $k_*(C)$  is determined by the parameters  $w(C)$  and  $R(C)$ : There exist absolute constants  $c_1, c_2 > 0$  such that

$$(2.9) \quad c_1 n \frac{w(C)^2}{R(C)^2} \leq k_*(C) \leq c_2 n \frac{w(C)^2}{R(C)^2}$$

for every symmetric convex body  $C$  in  $\mathbb{R}^n$ . The lower bound appears in Milman's proof of Dvoretzky's theorem (see [13]) and the upper bound was proved in [16]. The following Lemma is proved in [10]:

**Lemma 2.1.** *Let  $C$  be a symmetric convex body in  $\mathbb{R}^n$ . Then,*

- (i)  $w_q(C) \simeq w(C)$  for all  $q \leq k_*(C)$ .
- (ii)  $w_q(C) \simeq \sqrt{q/n} R(C)$  for all  $k_*(C) \leq q \leq n$ .
- (iii)  $w_q(C) \simeq R(C)$  for all  $q \geq n$ .

**2.5.** We define

$$(2.10) \quad q_*(\mu) := \max\{k \leq n : k_*(Z_k(\mu)) \geq k\}.$$

Then, the main result of [18] states that, for every centered log-concave probability measure  $\mu$  on  $\mathbb{R}^n$ , one has

$$(2.11) \quad I_{-q}(\mu) \simeq I_q(\mu)$$

for every  $1 \leq q \leq q_*(\mu)$ . In particular, for all  $q \leq q_*(\mu)$  one has  $I_q(\mu) \leq CI_2(\mu)$ , where  $C > 0$  is an absolute constant.

Assuming that  $\mu$  is isotropic, one can check that  $q_*(\mu) \geq c\sqrt{n}$ , where  $c > 0$  is an absolute constant (for a proof, see [17]).

### 3. $\Psi_\alpha$ -ESTIMATES FOR MARGINALS

Let  $\mu$  be an isotropic log-concave probability measure on  $\mathbb{R}^n$ . We first prove Theorem 1.1(i) and Theorem 1.2.

**3.1. The case  $k \leq \sqrt{n}$ .** From (2.2) we see that  $Z_q(\mu) \subseteq cqZ_2(\mu)$  for all  $q \geq 2$ . Since  $\mu$  is isotropic, we have  $Z_2(\mu) = B_2^n$ , and hence,  $R(Z_q(\mu)) \leq cq$  for all  $q \geq 1$ .

Let  $d(q) = n \frac{w^2(Z_q(\mu))}{R^2(Z_q(\mu))}$  and  $D(\mu) = \{q \geq 2 : q \leq d(q)\}$ . Let  $q_0$  be the maximum of the set of  $q \geq 2$  for which  $[2, q] \subseteq D(\mu)$ . Then, by the continuity of  $d(q)$ , we have  $q_0 = d(q_0)$ . In particular, from Lemma 2.1 we have

$$(3.1) \quad w(Z_{q_0}(\mu)) \simeq w_{q_0}(Z_{q_0}(\mu)) \simeq \sqrt{q_0/n} I_{q_0}(\mu) \geq c\sqrt{q_0}.$$

It follows that

$$(3.2) \quad q_0 = n \frac{w^2(Z_{q_0}(\mu))}{R^2(Z_{q_0}(\mu))} \geq \frac{cnq_0}{q_0^2} = \frac{cn}{q_0},$$

and hence  $q_0 \geq c\sqrt{n}$ . By the definition of  $q_0$ , for all  $q \leq c\sqrt{n}$  we have  $q \leq d(q)$ , and the previous argument, applied for  $q$ , shows that

$$(3.3) \quad w(Z_q(\mu)) \geq c\sqrt{q} \quad \text{and} \quad k_*(Z_q(\mu)) \geq cn/q.$$

Now, let  $k \leq \sqrt{n}$ . For every  $1 \leq q \leq k$  we have  $I_q(\mu) \leq CI_2(\mu) = C\sqrt{n}$ , and hence,

$$(3.4) \quad w(Z_q(\mu)) \leq w_q(Z_q(\mu)) \leq C\sqrt{q}.$$

Then, if we fix  $q \leq k$ , Dvoretzky's theorem (see [15]) shows that

$$(3.5) \quad P_F(Z_q(\mu)) \simeq w(Z_q(\mu))(B_2^n \cap F)$$

for all  $F$  in a subset  $B_{k,q}$  of  $G_{n,k}$  of measure

$$(3.6) \quad \nu_{n,k}(B_{k,q}) \geq 1 - e^{-ck_*(Z_q(\mu))} \geq 1 - e^{-c_1\sqrt{n}}.$$

Applying this argument for  $q = 2^i$ ,  $i = 1, \dots, \log_2 k$ , and taking into account the fact that  $Z_p(\mu) \subseteq Z_q(\mu) \subseteq cZ_p(\mu)$  if  $p < q \leq 2p$ , we conclude that there exists  $B_k \subset G_{n,k}$  with  $\nu_{n,k}(B_k) \geq 1 - e^{-c_2\sqrt{n}}$  such that, for every  $F \in B_k$  and every  $1 \leq q \leq k$ ,

$$(3.7) \quad Z_q(\pi_F(\mu)) = P_F(Z_q(\mu)) \simeq w(Z_q(\mu))(B_2^n \cap F).$$

From (3.3) and (3.4) we have  $w(Z_q(\mu)) \simeq \sqrt{q}$  for all  $q \leq \sqrt{n}$ . Therefore, the last formula can be written in the form

$$(3.8) \quad h_{Z_q(\pi_F(\mu))}(\theta) \simeq \sqrt{q}$$

for all  $F \in B_k$ ,  $\theta \in S_F$  and  $1 \leq q \leq k$ .

From the inequality

$$(3.9) \quad \sup_{1 \leq q \leq k} \frac{\|\langle \cdot, \theta \rangle\|_{L_q(\pi_F(\mu))}}{\sqrt{q}} = \sup_{1 \leq q \leq k} \frac{h_{Z_q(\pi_F(\mu))}(\theta)}{\sqrt{q}} \leq C, \quad \theta \in S_F$$

we immediately get Theorem 1.1(i).

Next, we give the proof of Theorem 1.2, following an argument which essentially appears in [19]. Since

$$(3.10) \quad h_{Z_{2q}(\pi_F(\mu))}(\theta) \simeq h_{Z_q(\pi_F(\mu))}(\theta),$$

using the Paley–Zygmund inequality we see that, for every  $q \geq 1$  and every  $\theta \in S_F$ ,

$$(3.11) \quad [\pi_F(\mu)] \left( \left\{ x \in F : |\langle x, \theta \rangle| \geq \frac{1}{2} h_{Z_q(\pi_F(\mu))}(\theta) \right\} \right) \geq e^{-q}.$$

Then, (3.8) gives

$$(3.12) \quad [\pi_F(\mu)] (\{x \in F : |\langle x, \theta \rangle| \geq c\sqrt{q}\}) \geq e^{-q}$$

for every  $1 \leq q \leq k$  and every  $\theta \in S_F$ .

If  $1 \leq t \leq c\sqrt{k}$  we can write  $t$  in the form  $t := \sqrt{q}$  for some  $q \leq k$ . Then, a direct application of (3.12) gives

$$(3.13) \quad [\pi_F(\mu)] (\{x \in \mathbb{R}^n : |\langle x, \theta \rangle| \geq t\}) \geq e^{-t^2/c}$$

for all  $\theta \in S_F$ . This implies that  $\overline{sg}_{\pi_F(\mu)}(\theta) \geq c$  for all  $\theta \in S_F$ , where  $c > 0$  is an absolute constant.  $\square$

It remains to prove Theorem 1.1(ii).

**3.2. The case  $k \gg \sqrt{n}$ .** Fix  $k = n^\delta$ , where  $\delta \in (\frac{1}{2}, 1)$ , and let  $1 \leq q \leq k$ . Then, using Lemma 2.1 and the identity (2.4), we see that, for every  $F \in G_{n,k}$ ,

$$(3.14) \quad R(Z_q(\pi_F(\mu))) \simeq c_1 w_k(Z_q(\pi_F(\mu))) = w_k(P_F(Z_q(\mu))).$$

Therefore,

$$\begin{aligned} \left( \int_{G_{n,k}} R^k(Z_q(\pi_F(\mu))) d\nu_{n,k}(F) \right)^{1/k} &\simeq \left( \int_{G_{n,k}} w_k^k(P_F(Z_q(\mu))) d\nu_{n,k}(F) \right)^{1/k} \\ &= \left( \int_{G_{n,k}} \int_{S_F} h_{P_F(Z_q(\mu))}^k(\theta) d\sigma_F(\theta) d\nu_{n,k}(F) \right)^{1/k}, \end{aligned}$$

where  $\sigma_F$  is the rotationally invariant probability measure on the sphere  $S_F := S^{n-1} \cap F$ . Since

$$(3.15) \quad h_{P_F(Z_q(\mu))}(\theta) = h_{Z_q(\mu)}(\theta), \quad \theta \in S_F,$$

and

$$(3.16) \quad \int_{G_{n,k}} \int_{S_F} h_{Z_q(\mu)}^k(\theta) d\sigma_F(\theta) d\nu_{n,k}(F) = \int_{S^{n-1}} h_{Z_q(\mu)}^k(\theta) d\sigma(\theta) = w_k^k(Z_q(\mu)),$$

we get the following.

**Lemma 3.1.** *Let  $\mu$  be an isotropic log-concave probability measure on  $\mathbb{R}^n$ . For every  $1 \leq k \leq n$  and  $1 \leq q \leq k$ ,*

$$(3.17) \quad \left( \int_{G_{n,k}} R^k(Z_q(\pi_F(\mu))) d\nu_{n,k}(F) \right)^{1/k} \simeq w_k(Z_q(\mu)).$$

The next Lemma gives some bounds for  $w_k(Z_q(\mu))$ .

**Lemma 3.2.** *Let  $\mu$  be an isotropic log-concave probability measure on  $\mathbb{R}^n$ . If  $k = n^\delta$ ,  $\delta \in (\frac{1}{2}, 1)$  and  $1 \leq q \leq k$ , then*

$$(3.18) \quad w_k(Z_q(\mu)) \leq c_3 q^{1/\alpha(\delta)},$$

where  $\alpha(\delta) = \frac{2\delta}{3\delta-1}$ .

*Proof.* Let  $1 \leq q \leq k$ . We distinguish two cases:

(i) Assume that  $k \leq n/q$ . Then, we have  $q \leq n/q$  and (3.3) shows that  $k_*(Z_q(\mu)) \geq cn/q$ . Therefore,  $k \leq ck_*(Z_q(\mu))$ , and hence,

$$(3.19) \quad w_k(Z_q(\mu)) \simeq w(Z_q(\mu)) \leq w_q(Z_q(\mu)) \simeq \sqrt{q}.$$

(ii) Assume that  $k > n/q$ . From Lemma 2.1 we have that  $w_k(Z_q(\mu)) \simeq w(Z_q(\mu))$  if  $k \leq k_*(Z_q(\mu))$  and  $w_k(Z_q(\mu)) \simeq \sqrt{k/n} R(Z_q(K))$  if  $k \geq k_*(Z_q(\mu))$ . Since  $q \leq k$ , using (3.4) we get that  $w_k(Z_q(K)) \leq f(q, k)$ , where  $f(q, k) \leq c\sqrt{q}$  if  $q \leq k \leq k_*(Z_q(\mu))$  and  $f(q, k) \leq cq\sqrt{k/n}$  if  $k \geq k_*(Z_q(\mu))$ . Note that  $k_*(Z_q(\mu)) \geq n/k$ . So we get that

$$(3.20) \quad f(q, k) \leq c\sqrt{q} \text{ if } q \leq n/k \text{ and } f(q, k) \leq q\sqrt{k/n} \text{ if } n/k \leq q.$$

We want  $q^{1-\frac{1}{\alpha}} \sqrt{k} \ll \sqrt{n}$  for all  $q \leq k$ . This will be true if  $k^{\frac{3}{2}-\frac{1}{\alpha}} \simeq n^{1/2}$ . Since  $k = n^\delta$ , the optimal value of  $\alpha$  is

$$(3.21) \quad \alpha(\delta) = \frac{2\delta}{3\delta-1}.$$

From (i) we check that (3.18) holds true for  $k \leq n/q$  as well. This proves the Lemma.  $\square$

*Proof of Theorem 1.1(ii).* We apply Markov's inequality for  $q = 2^i$ ,  $i = 1, \dots, \log_2 k$  in Lemma 3.1, and taking into account the fact that  $Z_p(\mu) \subseteq Z_q(\mu) \subseteq cZ_p(\mu)$  if  $p < q \leq 2p$ , we conclude that

$$(3.22) \quad \sup_{1 \leq q \leq k} \frac{R(Z_q(\pi_F(\mu)))}{w_k(Z_q(\mu))} \leq C,$$

where  $C > 0$  is an absolute constant, for all  $F$  in a subset  $A_k$  of  $G_{n,k}$  with measure  $\nu_{n,k}(A_k) \geq 1 - (\log_2 k)e^{-2k} \geq 1 - e^{-k}$ .

Now, we are using the estimates from Lemma 3.2; for every  $F \in A_k$  we have

$$(3.23) \quad \|\langle \cdot, \theta \rangle\|_{\psi'_{\alpha(\delta)}} = \sup_{1 \leq q \leq k} \frac{\|\langle \cdot, \theta \rangle\|_q}{q^{1/\alpha(\delta)}} \leq C_1 \sup_{1 \leq q \leq k} \frac{R(Z_q(\pi_F(\mu)))}{w_k(Z_q(\mu))} \leq C_2$$

for all  $\theta \in S_F$ , where  $C_2 > 0$  is an absolute constant.  $\square$

#### 4. FURTHER REMARKS AND APPLICATIONS

**4.1.** Assume that  $\mu$  is a  $\psi_\beta$  measure with constant  $r > 0$  for some  $\beta \in (1, 2)$ . Then, the argument of Section 3 leads to the following generalization of Theorem 1.1.

**Theorem 4.1.** *Let  $\beta \in (1, 2)$  and let  $\mu$  be an isotropic log-concave probability measure on  $\mathbb{R}^n$  which is  $\psi_\beta$  with constant  $r > 0$ .*

- (i) *If  $k \leq n^{\frac{\beta}{2}}$  then there exists  $A_k \subseteq G_{n,k}$  with measure  $\nu_{n,k}(A_k) > 1 - \exp(-cn^{\frac{\beta}{2}})$  such that, for every  $F \in A_k$ ,  $\pi_F(\mu)$  is a  $\psi'_2$ -measure with constant  $C(r)$ .*
- (ii) *If  $k = n^\delta$ ,  $\frac{\beta}{2} < \delta < 1$  then there exists  $A_k \subseteq G_{n,k}$  with measure  $\nu_{n,k}(A_k) > 1 - \exp(-ck)$  such that, for every  $F \in A_k$ ,  $\pi_F(\mu)$  is a  $\psi'_{\alpha(\delta, \beta)}$ -measure with constant  $C(r)$ , where  $\alpha(\delta, \beta) = \frac{2\beta\delta}{(2\delta - \beta) + \beta\delta}$ .*

**4.2.** The estimate for  $\alpha(\delta, \beta)$  in Theorem 4.1 is optimal in the following sense: let  $\mu$  be an isotropic log-concave probability measure on  $\mathbb{R}^n$  which is  $\psi_\beta$  and has the property that there exists  $\theta \in S^{n-1}$  such that  $h_{Z_q(\mu)}(\theta) \simeq q^{\frac{1}{\beta}}$  for all  $1 \leq q \leq n$ . Then, Lemma 2.1 shows that for  $k = n^\delta \geq n^{\frac{\beta}{2}}$  we have

$$(4.1) \quad w_k(Z_k(\mu)) \simeq \sqrt{k/n} R(Z_k(\mu)) \simeq \sqrt{k/n} k^{\frac{1}{\beta}} = k^{\frac{1}{\alpha(\beta, \delta)}}.$$

Then, using (3.17) and the Paley-Zygmund inequality we can check that there exists  $A_k \subseteq G_{n,k}$  with measure  $\nu_{n,k}(A_k) > \exp(-ck)$  such that, for every  $F \in A_k$  there exists  $\theta \in S_F$  such that  $h_{Z_k(\mu)}(\theta) \geq ck^{\frac{1}{\alpha(\beta, \delta)}}$ .

**4.3.** For every  $p \geq 1$  we consider the convex body  $K_p(\mu)$  (introduced by K. Ball in [1]) with gauge function

$$(4.2) \quad \|x\|_{K_p(\mu)} := \left( \frac{p}{f_\mu(0)} \int_0^\infty f_\mu(rx) r^{p-1} dr \right)^{-1/p}.$$

Let  $1 \leq k < n$  and  $F \in G_{n,k}$ . For  $\theta \in S_F$  we define

$$(4.3) \quad \|\theta\|_{B_{k+1}(\mu, F)} := \|\theta\|_{K_{k+1}(\pi_F(\mu))}.$$

For all  $1 \leq q \leq k < n$  and  $F \in G_{n,k}$ , one has (see [18] and [3])

$$(4.4) \quad f_{\pi_F(\mu)}(0)^{\frac{1}{k}} Z_q(\pi_F(\mu)) \simeq f_\mu(0)^{\frac{1}{n}} Z_q(\widetilde{B_{k+1}(\mu, F)}).$$

If  $\mu$  is isotropic, then  $\widetilde{B}_{k+1}(\mu, F)$  is an isotropic convex body in  $F$ . In particular, the case  $p = 2$  of (4.3) shows that

$$(4.5) \quad f_{\pi_F(\mu)}(0)^{\frac{1}{k}} \simeq f_\mu(0)^{\frac{1}{n}} L_{B_{k+1}(\mu, F)}.$$

Since the  $\psi_\alpha$  and  $\psi'_\alpha$  norms are equivalent for convex bodies, as an immediate consequence of the above formulas we get the following version of Theorem 4.1:

**Theorem 4.2.** *Let  $\beta \in (1, 2)$  and let  $\mu$  be an isotropic log-concave probability measure on  $\mathbb{R}^n$  which is  $\psi_\beta$  with constant  $r > 0$ .*

- (i) *If  $k \leq n^{\frac{\beta}{2}}$  then there exists  $A_k \subseteq G_{n,k}$  with measure  $\nu_{n,k}(A_k) > 1 - \exp(-cn^{\frac{\beta}{2}})$  such that, for every  $F \in A_k$ ,  $\widetilde{B}_{k+1}(\mu, F)$  is a  $\psi_2$ -body with constant  $C(r)$ .*
- (ii) *If  $k = n^\delta$ ,  $\frac{\beta}{2} < \delta < 1$  then there exists  $A_k \subseteq G_{n,k}$  with measure  $\nu_{n,k}(A_k) > 1 - \exp(-ck)$  such that, for every  $F \in A_k$ ,  $\widetilde{B}_{k+1}(\mu, F)$  is a  $\psi_{\alpha(\delta, \beta)}$ -body with constant  $C(r)$ , where  $\alpha(\delta, \beta) = \frac{2\beta\delta}{(2\delta - \beta) + \beta\delta}$ .*

**4.4.** It was mentioned in §2.5 that if  $\mu$  is an isotropic log-concave probability measure on  $\mathbb{R}^n$ , then  $I_{-q}(\mu) \simeq I_q(\mu)$  for every  $1 \leq q \leq q_*(\mu)$ . If  $\mu$  is a  $\psi_\beta$ -measure, then  $q_*(\mu) \geq cn^{\frac{\beta}{2}}$ . This gives the lower bound

$$(4.6) \quad I_{-q}(\mu) \geq c\sqrt{n}$$

for all  $q \leq n^{\frac{\beta}{2}}$ . Using the results of this note, we can give some non-trivial lower bounds for  $I_{-q}(\mu)$  when  $q \gg n^{\frac{\beta}{2}}$ . Let  $f$  be the density of  $\mu$ . We start with a formula from [18, Proposition 4.6]: taking into account (4.5) we see that, for every  $1 \leq k < n$ ,

$$(4.7) \quad I_{-k}(\mu) \simeq \sqrt{n} \left( \int_{G_{n,k}} L_{B_{k+1}(\mu, F)}^k d\nu_{n,k}(F) \right)^{-\frac{1}{k}}.$$

Then, what we need is an upper bound for the quantity

$$(4.8) \quad \int_{G_{n,k}} L_{B_{k+1}(\mu, F)}^k d\nu_{n,k}(F)$$

in the case  $k = n^\delta$ ,  $\delta \in \left(\frac{\beta}{2}, 1\right)$ . We now use the following fact (see [6, Theorem 2.5.4]): If  $\alpha \in (1, 2]$  and  $C$  is an isotropic convex body in  $\mathbb{R}^k$  which is  $\psi_\alpha$  with constant  $r > 0$ , then

$$(4.9) \quad L_C \leq cr^{\frac{\alpha}{2}} k^{\frac{2-\alpha}{4}} \log k.$$

From Lemma 3.2 we know that, for every  $1 \leq q \leq k$ , we have  $w_k(Z_q(\mu)) \leq cq^{1/\alpha_*}$ , where  $\alpha_* = \frac{2\beta\delta}{(2\delta - \beta) + \beta\delta}$ .

Then, the argument of Lemma 3.1 shows that the probability that  $R(Z_q(B_{k+1}(\mu, F))) > csq^{1/\alpha_*}$  is less than  $s^{-k}$ . It follows that, for every  $s \geq 1$  we have

$$(4.10) \quad \sup_{\theta \in S_F} \|\langle \cdot, \theta \rangle\|_{\psi_{\alpha_*}(B_{k+1}(\mu, F))} \leq c_1 s$$

on a subset  $B_{k,s}$  of  $G_{n,k}$  of measure  $\nu_{n,k}(B_{k,s}) \geq 1 - s^{-k}$ . Therefore,

$$(4.11) \quad L_{B_{k+1}(\mu, F)} \leq c_2 s^{\frac{\alpha_*}{2}} k^{\frac{2-\alpha_*}{4}} \log k$$

for all  $F \in B_{k,s}$ . Set  $m(k) = c_2 k^{\frac{2-\alpha_*}{4}} \log k$ . Then, we can estimate the integral (4.7) as follows:

$$\begin{aligned}
\int_{G_{n,k}} L_{B_{k+1}(\mu,F)}^k d\nu_{n,k}(F) &= \int_0^\infty kt^{k-1} \nu_{n,k}(F : L_{B_{k+1}(\mu,F)} \geq t) dt \\
&\leq m^k(k) + \int_{m(k)}^\infty kt^{k-1} \nu_{n,k}(F : L_{B_{k+1}(\mu,F)} \geq t) dt \\
&= m^k(k) \left[ 1 + \int_1^\infty ks^{\frac{(k-1)\alpha_*}{2}} \nu_{n,k}(F : L_{B_{k+1}(\mu,F)} \geq m(k)s^{\frac{\alpha_*}{2}}) ds \right] \\
&\leq m^k(k) \left[ 1 + \frac{k\alpha_*}{2} \int_1^\infty s^{\frac{(k-1)\alpha_*}{2}} s^{\frac{\alpha_*}{2}-1} s^{-k} ds \right] \\
&= m^k(k) \left[ 1 + \frac{k\alpha_*}{2} \int_1^\infty s^{-1-k(1-\frac{\alpha_*}{2})} ds \right] \\
&\simeq m^k(k) \simeq \left( k^{\frac{2-\alpha_*}{4}} \log k \right)^k.
\end{aligned}$$

Inserting this information into (4.7) we get

$$(4.12) \quad I_{-k}(\mu) \geq \frac{c\sqrt{n}}{k^{\frac{2-\alpha_*}{4}} \log k} \geq \frac{c}{\log n} n^{\frac{1}{2} - \frac{\delta(2-\alpha_*)}{4}}.$$

Using our estimate for  $\alpha_* = \alpha(\delta, \beta)$ , we finally get the following:

**Theorem 4.3.** *Let  $\beta \in [1, 2]$  and let  $\mu$  be an isotropic log-concave probability measure on  $\mathbb{R}^n$ , which is a  $\psi_\beta$ -measure with constant  $r > 0$ .*

- (1) *If  $k \leq n^{\frac{\beta}{2}}$ , then  $I_{-k}(\mu) \geq c(r)\sqrt{n}$ .*
- (2) *If  $k = n^\delta$  for some  $\delta \in (\frac{\beta}{2}, 1)$ , then*

$$(*) \quad I_{-k}(\mu) \geq c(r) \frac{n^{\frac{(1-\delta)(2\delta-\beta)+\beta\delta}{2[(2\delta-\beta)+\beta\delta]}}}{\log n}.$$

**Final remark.** In Theorem 4.3, we can actually obtain a stronger estimate. For an isotropic convex body  $C$  in  $\mathbb{R}^s$ , let  $C_1 = C \cap (4\sqrt{s}L_C)B_2^s$  and  $\overline{C} = \widetilde{C}_1$ . For any  $F \in G_{n,k}$  we consider the body  $\overline{B_{k+1}(K,F)}$  and, using the estimates from Lemma 3.2, we observe that

- (i)  $h_{Z_q(\overline{B_{k+1}(K,F)})}(\theta) \leq c\sqrt{q}L_{B_{k+1}(K,F)}$ , for  $1 \leq q \leq \left(\frac{n}{k}\right)^\beta$ ,
- (ii)  $h_{Z_q(\overline{B_{k+1}(K,F)})}(\theta) \leq c\sqrt{\frac{k}{n}}q^{\frac{1}{\beta}}L_{B_{k+1}(K,F)}$ , for  $\left(\frac{n}{k}\right)^\beta \leq q \leq n^{\frac{\beta}{2}}$ ,
- (iii)  $h_{Z_q(\overline{B_{k+1}(K,F)})}(\theta) \leq c\sqrt{k}L_{B_{k+1}(K,F)}$ , for  $n^{\frac{\beta}{2}} \leq q \leq k$ .

This implies that  $\overline{B_{k+1}(K,F)}$  is a  $\psi_2$ -body with constant  $O(n^{\frac{2\delta-\beta}{4}})$ . Inserting this information in the proof of Theorem 4.3, and using the fact – proved in [3] – that if  $C$  is a  $\psi_2$  body with constant  $r$  then  $L_C \leq cr\sqrt{\log(er)}$  in the place of (4.9), one can prove the following fact: Let  $\beta \in [1, 2]$  and let  $\mu$  be an isotropic log-concave probability measure on  $\mathbb{R}^n$ , which is a  $\psi_\beta$ -measure with constant  $r > 0$ .

- (i) *If  $k \leq n^{\frac{\beta}{2}}$ , then  $I_{-k}(\mu) \geq cr\sqrt{n}$ .*

(ii) If  $k = n^\delta$  for some  $\delta \in \left(\frac{\beta}{2}, 1\right)$ , then

$$(**) \quad I_{-k}(\mu) \geq cr \frac{n^{\frac{1}{2} - \frac{2\delta - \beta}{4}}}{\sqrt{\log\left((crn)^{\frac{2\delta - \beta}{4}}\right)}}.$$

Using this result, we can also slightly improve the small probability estimate

$$\mu\left(\{x \in \mathbb{R}^n : \|x\|_2 \leq c\varepsilon\sqrt{n}\}\right) \leq \varepsilon^{\sqrt{n}}$$

from [18]. Using (\*\*) one can show that if  $\mu$  is an isotropic *log*-concave measure in  $\mathbb{R}^n$  then, for every  $\varepsilon \in (0, 1)$ ,

$$(4.13) \quad \mu\left(\{x \in \mathbb{R}^n : \|x\|_2 \leq c\varepsilon\sqrt{n}\}\right) \leq \varepsilon^{\sqrt{n}} \min\left\{1, \varepsilon^{n^{\delta(\varepsilon, n)}}\right\},$$

where  $\delta(\varepsilon, n) = \frac{\log(\varepsilon^{-2})}{\log n} - \log \log n$ . We omit the detailed proofs of these assertions; we would also like to mention that these estimates are optimal up to our current knowledge on  $L_K$  and a logarithmic in the dimension term.

#### REFERENCES

1. K. M. Ball, *Logarithmically concave functions and sections of convex sets in  $\mathbb{R}^n$* , *Studia Math.* **88** (1988), 69–84.
2. C. Borell, *Convex set functions in  $d$ -space*, *Period. Math. Hungar.* **6** (1975) 111–136.
3. N. Dafnis and G. Paouris, *Small ball probability estimates,  $\psi_2$ -behavior and the hyperplane conjecture*, *Journal of Functional Analysis* **258** (2010), 1933–1964.
4. B. Fleury, *Concentration in a thin Euclidean shell for log-concave measures*, Preprint.
5. B. Fleury, O. Guédon and G. Paouris, *A stability result for mean width of  $L_p$ -centroid bodies*, *Advances in Mathematics* **214** (2007), 865–877.
6. A. Giannopoulos, *Notes on isotropic convex bodies*, *Warsaw University Notes* (2003).
7. B. Klartag, *A central limit theorem for convex sets*, *Invent. Math.* **168** (2007), 91–131.
8. B. Klartag, *Power-law estimates for the central limit theorem for convex sets*, *J. Funct. Analysis* **245** (2007), 284–310.
9. B. Klartag, *On nearly radial marginals of high-dimensional probability measures*, *J. Eur. Math. Soc.* **12** (2010), 723–754.
10. A. Litvak, V. D. Milman and G. Schechtman, *Averages of norms and quasi-norms*, *Math. Ann.* **312** (1998), 95–124.
11. E. Lutwak and G. Zhang, *Blaschke-Santaló inequalities*, *J. Differential Geom.* **47** (1997), 1–16.
12. E. Lutwak, D. Yang and G. Zhang,  *$L^p$  affine isoperimetric inequalities*, *J. Differential Geom.* **56** (2000), 111–132.
13. V. D. Milman, *A new proof of A. Dvoretzky's theorem in cross-sections of convex bodies*, (Russian), *Funkcional. Anal. i Priložen.* **5** (1971) 28–37.
14. V. D. Milman and A. Pajor, *Isotropic positions and inertia ellipsoids and zonoids of the unit ball of a normed  $n$ -dimensional space*, *GAFA Seminar 87-89*, *Springer Lecture Notes in Math.* **1376** (1989), 64–104.
15. V. D. Milman and G. Schechtman, *Asymptotic Theory of Finite Dimensional Normed Spaces*, *Lecture Notes in Math.* **1200** (1986), Springer, Berlin.
16. V. D. Milman and G. Schechtman, *Global versus Local asymptotic theories of finite-dimensional normed spaces*, *Duke Math. Journal* **90** (1997), 73–93.
17. G. Paouris, *Concentration of mass on convex bodies*, *Geometric and Functional Analysis* **16** (2006), 1021–1049.
18. G. Paouris, *Small ball probability estimates for log-concave measures*, *Trans. Amer. Math. Soc.* (to appear).
19. G. Paouris, *On the existence of supergaussian directions on convex bodies*, Preprint.
20. G. Pisier, *The Volume of Convex Bodies and Banach Space Geometry*, *Cambridge Tracts in Mathematics* **94** (1989).

21. P. Pivovarov, *On the volume of caps and bounding the mean-width of an isotropic convex body*, Math. Proc. Cambridge Philos. Soc. (to appear).

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