Construction of suitable weak solutions for the 3D incompressible NSEs

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Outline

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1. BASIC FACTS ABOUT THE 3D NSEs
2. SUITABLE WEAK SOLUTIONS
3. GALERKIN IN TORUS

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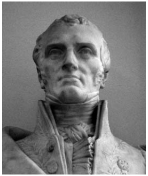
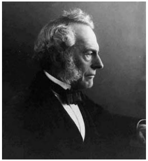
BASIC FACTS ABOUT THE NSE



Claude Louis Marie Henri Navier



George Gabriel Stokes

1. BASIC FACTS ABOUT THE 3D NSEs
2. SUITABLE WEAK SOLUTIONS
3. GALERKIN IN TORUS
4. GALERKIN + DIRICHLET
5. FOURIER IN TORUS

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- u_0 is the initial data.
- f a source term.
- ρ is chosen equal to unity.
- ν is viscosity (inverse of Reynolds number).

EXISTENCE

- J. Leray (1934): introduces the notion of turbulent solution. A turbulent solution is a weak solution in $u \in L^2(0,T;\mathbf{H}^1(\Omega)) \cap L^\infty(0,T;\mathbf{L}^2(\Omega))$ + global energy inequality:

$$\frac{1}{2}\|u\|^2_{\mathbf{L}^2} + \nu \int_0^T \|\nabla u\|^2_{L^2} dt \leq \frac{1}{2}\|u_0\|^2_{\mathbf{L}^2} + \int_0^T \int_\Omega f \cdot u dx dt$$

EXISTENCE

- J. Leray uses mollification to prove existence:
  $\psi \in D(\mathbb{R}^3)$, $\psi \geq 0$, $\int_{\mathbb{R}^3} \psi = 1$, $\psi_\epsilon(x) = \frac{1}{\epsilon}\psi\left(\frac{x}{\epsilon}\right)$.

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- Uniqueness in more regular classes (with short time existence, Prodi-Serrin (1959-1963))

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  $\Rightarrow$ evolution under NSE: $B^{-1}_{\infty,\infty} \to C([0,T]; B^{-1}_{\infty,\infty})$ is not continuous.
- Cheskidov, Shvydkoy (Arxiv, 2009): The evolution under NSE $H^s \cap B^{-1}_{\infty,\infty} \to C([0,T]; B^{-1}_{\infty,\infty})$ is not continuous, $\forall s < \frac{1}{2}$.

SUITABLE WEAK SOLUTION

Definition (V. Scheffer (1976))

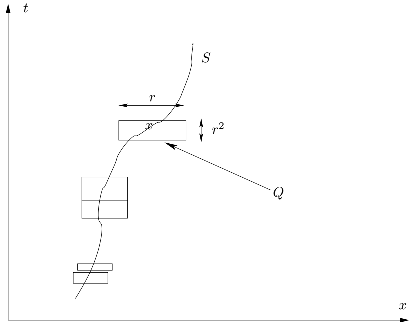
A NS weak solution is said to be suitable weak solutions iff (u, p) is a weak solution and

$$\partial_t(\tfrac{1}{2}u^2) + \nabla\cdot(u(\tfrac{1}{2}u^2 + p)) - \nu\nabla^2(\tfrac{1}{2}u^2) + \nu(\nabla u)^2 - f \cdot u \leq 0.$$

in $\mathcal{D}'((0, T)\times\Omega)$

SUITABLE WEAK SOLUTION

- Singular set: S
  $S = \{(x,t) \in \Omega \times]0,T[\;:\; u \notin L^\infty(V), \forall V \text{ s.t. } (x,t) \in V\}$



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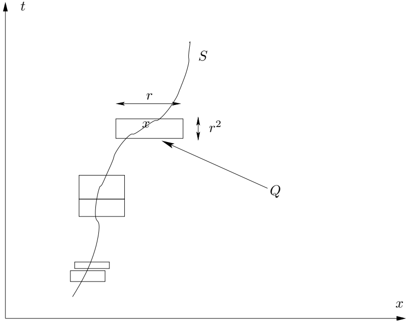

- $\mathcal{P}^1(S) = \lim_{\delta\to 0^+} \inf\{\sum r_i^{1}, \ : \ S \subset \cup Q(M_i,r_i), \ r_i < \delta\}$

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$\Rightarrow$ Singularities (if any) of suitable weak solutions are almost pointwise in space/time.

$\boxed{\text{Best partial regularity theorem to date.}}$

CONSTRUCTION OF SUITABLE SOLUTIONS

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Theorem (Leray (1934), Duchon-Robert (2000))

Unique weak solution for all $t > 0$, and $u_\epsilon \underset{\epsilon\to 0}{\longrightarrow} u$ (up to subsequences) and u is suitable.

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- Let $P_\varepsilon : \dot{\mathbf{L}}^2(\Omega) \longrightarrow \mathbf{X}_\varepsilon$ be L^2-projection.

$$\dot{\mathbf{L}}^2(\Omega) = \mathbf{X}_\varepsilon \oplus (\mathbf{X}_\varepsilon)^\perp$$

EX 2: CONSTRUCTION BY MOLLIFICATION/NLGM

- Solve for u_ϵ, p_ϵ s.t.

$$\begin{cases} \partial_t(P_\varepsilon u_\epsilon) + P_\varepsilon u_\epsilon\cdot\nabla u_\epsilon + \nabla p_\epsilon - \nu\nabla^2 u_\epsilon = \mathbf{f} \quad \text{in } \Omega \\ \nabla\cdot u_\epsilon = 0 \quad \text{in } \Omega, \\ u_\epsilon \text{ is periodic}, \quad u_\epsilon|_{t=0} = u_0, \end{cases}$$

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Theorem

For all $\epsilon > 0$, problem is well-posed (existence + uniqueness). $u_\epsilon \to u$, $p_\epsilon \to p$ as $\epsilon \to 0$ (in appropriate spaces, up to subsequences), u and p are suitable weak solution to N.S.

EX 3: CONSTRUCTION BY HYPERVISCOSITY

- Add vanishing hyperviscosity (Lions (1959)). Ω is the d-torus, d is the space dimension.

$$\begin{cases} \partial_t u_\epsilon + u_\epsilon \cdot \nabla u_\epsilon + \nabla p_\epsilon - \nu \nabla^2 u_\epsilon + \varepsilon^{2\alpha}(-\nabla^2)^\alpha u_\epsilon = f, \\ \nabla \cdot u_\epsilon = 0 \\ u_\epsilon \text{ is periodic,} \\ u_\epsilon\big|_{t=0} = u_0. \end{cases}$$

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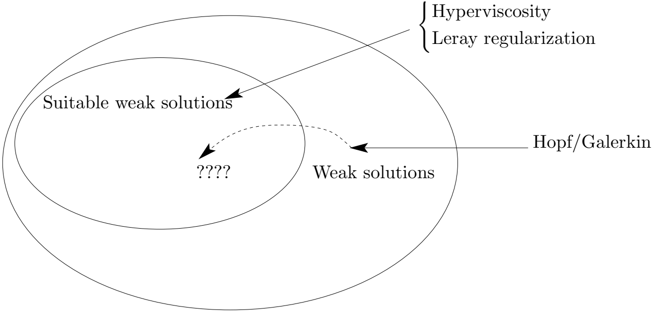
Theorem (Lions (1959), Beirão da Veiga (1985))

Unique weak solution for all $t > 0$ if $\alpha > \frac{d+2}{4}$, and $u_\epsilon \underset{\epsilon \to 0}{\longrightarrow} u$ (up to subsequences) and u is suitable.

MORE TRACTABLE QUESTIONS



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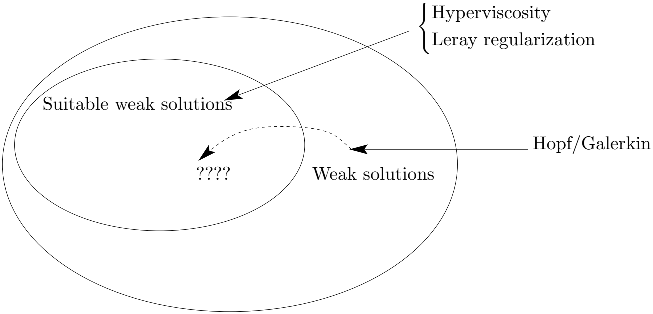

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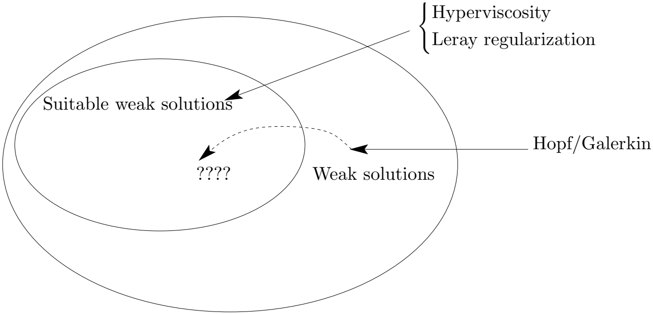

Q1: Is the set of suitable solutions a proper subset of weak solutions?

Q2: Do the Galerkin solutions end up to be suitable after all?

GALERKIN APPROX IN TORUS

Boris Grigorievich Galerkin

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- Assume there is $c > 0$ independent of h such that

$$\forall q_h \in M_h, \qquad \sup_{0\neq v_h \in X_h} \frac{(\nabla q_h, v_h)}{\|v_h\|_{\mathbf{L}^2}} \geq c\|\nabla q_h\|_{\mathbf{L}^2}.$$

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- Modify the nonlinear term as follows:

$$b_h(u,v,v) = \begin{cases} (u\cdot\nabla u + \frac{1}{2}u\nabla\cdot u, v) & \text{(Temam, 1967)} \\ ((\nabla\times u)\times u + \frac{1}{2}\nabla(\mathcal{K}_h(u^2)), v) & \end{cases}$$

where $\mathcal{K}_h : L^2(\Omega) \longrightarrow M_h$, linear L^2-stable approximation operator.

HYPOTHESES/DEFINITIONS (ctd.)

Definition (Discrete commutator property)

There is an operator $P_h \in \mathcal{L}(H^1_\#(\Omega); X_h)$ such that for all ϕ in $W^{2,\infty}_\#(\Omega)$ and all $v_h \in X_h$

$$\|\phi v_h - P_h(\phi v_h)\|_{H^l} \le c\, h^{1+m-l} \|v_h\|_{H^m} \|\phi\|_{W^{m+1,\infty}}, \quad 0 \le l \le m \le 1.$$

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- Similar hypothesis on pressure space M_h.
- Define $\Phi : \psi \longmapsto \phi\psi$, then

$$P_h(\phi v_h) - \phi v_h = P_h(\Phi(v_h)) - \Phi(P_h(v_h)) := [P_h, \Phi](v_h)$$

HYPOTHESES/DEFINITIONS (ctd.)

- For instance we want: $\|[P_h,\Phi]v_h\|_{L^2} \le c\,h\,\|v_h\|_{L^2}\|\phi\|_{W^{1,\infty}}$, i.e. $\|[P_h,\Phi]\|_{\mathcal{L}(\mathbb{P}_N\cap L^2,L^2)} \le c\,h\|\phi\|_{W^{1,\infty}}$.

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- FE and wavelet-based approximation spaces have the discrete commutator property

HYPOTHESES/DEFINITIONS (ctd.)

- Localization is the key property:

$$\|\psi - P_h\psi\|_{L^2(K)} \le c h_K \|\nabla\psi\|_{L^2(\Delta_K)}$$

K is a cell, $h_K := \mathrm{diam}(K)$, Δ_K is a patch containing K, $\mathrm{diam}(\Delta_K) \le c' h_K$.

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- Second key: P_h is a projection

$$\begin{aligned}\phi v_h - P_h(\phi v_h) &\approx \phi v_h - P_h((\phi(x_K) + \mathcal{O}(h_K))v_h) \\ &= \phi v_h - \phi(x_K)P_h(v_h) + \mathcal{O}(h_K)v_h \\ &= (\phi - \phi(x_K))v_h + \mathcal{O}(h_K) = \mathcal{O}(h_K)v_h\end{aligned}$$

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- Fourier-based approximation spaces do not have the discrete commutator property (No localization for interpolation properties).

GALERKIN FORMULATION

- Seek $u_h \in \mathcal{C}^1([0,T]; X_h)$ and $p_h \in \mathcal{C}^0([0,T]; M_h)$ such that for all $v_h \in X_h$, all $q_h \in M_h$, and all $t \in [0,T]$

$$\begin{cases} (\partial_t u_h, v) + b_h(u_h, u_h, v) - (p_h, \nabla\cdot v) + \nu(\nabla u_h, \nabla v) = \langle f, v\rangle, \\ (\nabla\cdot u_h, q) = 0, \\ u_h|_{t=0} = \mathcal{I}_h u_0, \end{cases}$$

where $\mathcal{I}_h : L^2(\Omega) \longrightarrow V_h$, L^2-stable interpolation operator.

THE MAIN RESULT

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Under the above hypotheses, if X_h and M_h have the discrete commutator property, the couple (u_h, p_h) convergences to a suitable solution to NS.

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- Question was open since Scheffer (1977).

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Question: Does the result hold for $\boxed{\text{Dirichlet BCs}}$?

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$$\begin{aligned}(u_h{\cdot}\nabla u_h, P_h(\phi u_h)) &= (u_h{\cdot}\nabla u_h, \phi u_h) + (u_h{\cdot}\nabla u_h, P_h(\phi u_h) - \phi u_h)\\ &= (u_h{\cdot}\nabla(\tfrac{1}{2}u_h^2), \phi) + (u_h{\cdot}\nabla u_h, [P_h, \Phi]u_h)\\ &\approx (\tfrac{1}{2}\nabla{\cdot}(u_h^2 u_h), \phi) + (u_h{\cdot}\nabla u_h, [P_h, \Phi]u_h)\\ &= -(\tfrac{1}{2}u_h^2 u_h, \nabla\phi) + (u_h{\cdot}\nabla u_h, [P_h, \Phi]u_h).\end{aligned}$$

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- Use the discrete commutator property to pass to the limit.

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- Use the discrete commutator property to pass to the limit.
- Idem form the term $u_h p_h$

OUTLINE

Johann Peter Gustav
Lejeune Dirichlet

HYPOTHESES/DEFINITIONS

- Finite element spaces, $X_h \subset \mathbf{H}^1_\#(\Omega)$ for velocity and $M_h \subset H^1_\#(\Omega)$ for pressure.

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- Modify the nonlinear term as follows:

$$b_h(u,v,v) = \begin{cases} (u{\cdot}\nabla u + \frac{1}{2} u \nabla{\cdot} u, v) & \text{(Temam, 1967)} \\ ((\nabla \times u)\times u + \frac{1}{2}\nabla(\mathcal{K}_h(u^2)), v) \end{cases}$$

where $\mathcal{K}_h : L^2(\Omega) \longrightarrow M_h$, linear L^2-stable approximation operator.

HYPOTHESES/DEFINITIONS (ctd.)

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PRELIMINARY RESULTS

Lemma

$\exists c_l > 0$ (non-increasing function) $\exists c_u > 0$ (non-decreasing function), independent of h:

$$c_l(|s|)\|v_h\|_{\widetilde{\mathbf{H}}_0^s} \le \|v_h\|_{\mathbf{V}_h^s} \le c_u(|s|)\|v_h\|_{\widetilde{\mathbf{H}}_0^s}, \quad \begin{cases} -\frac{1}{2} < s < \frac{3}{2}, & \textit{lower}, \\ -\frac{3}{2} < s < \frac{3}{2}, & \textit{upper} \end{cases}$$

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and for all $s \in (-\frac{3}{2}, 0]$

$$c_l(|s|)\|\nabla^2_h v_h\|_{\widetilde{\mathbf{H}}^s_0} \leq \|A_h v_h\|_{\mathbf{V}^s_h} \leq c_u(|s|)\|\nabla^2_h v_h\|_{\widetilde{\mathbf{H}}^s_0}, \quad \forall v_h \in \mathbf{V}_h.$$

GALERKIN FORMULATION

- Seek $u_h \in \mathcal{C}^1([0,T];X_h)$ and $p_h \in \mathcal{C}^0([0,T];M_h)$ such that for all $v_h \in X_h$, all $q_h \in M_h$, and all $t \in [0,T]$

  $$\begin{cases} (\partial_t u_h, v) + b_h(u_h,u_h,v) - (p_h, \nabla\cdot v) + \nu(\nabla u_h, \nabla v) = \langle f, v\rangle, \\ (\nabla\cdot u_h, q) = 0, \\ u_h|_{t=0} = \mathcal{I}_h u_0, \end{cases}$$

  where $\mathcal{I}_h : L^2(\Omega) \longrightarrow V_h$, L^2-stable interpolation operator.

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Theorem (Guermond (2007))

Under the above hypotheses, if X_h and M_h have the discrete commutator property, the couple (u_h, p_h) convergences to a suitable solution to NS.

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$\Rightarrow$ Galerkin solutions are suitable (provided discrete commutator property)

$\Rightarrow$ Hopf and Leray solutions are suitable

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- Alternative: (control on $\partial_t u_h$ and $\nabla_h^2 u_h$) + (Inf-sup) $\Rightarrow$ (control on p_h).
- Key: Use Fourier in time to derive estimates on $\partial_t u_h$ in negative norms.

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There is c independent of h so that,

$$\|\partial_t u_h\|_{H^{\tau-1}((0,T);\mathbf{H}^{-\alpha}(\Omega))} + \|u_h\|_{H^{\tau}((0,T);\mathbf{H}^{-\alpha}(\Omega))} \le c,$$

for all $\alpha \in [\frac{1}{4}, \frac{1}{2})$ *and for all* $\tau < \overline{\tau} := \frac{2}{5}(1+\alpha)$.

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- Slight improvement over Sohr and von Wahl (1986)

$$\text{SW (1986)} \Rightarrow \partial u_t \in H^{-\frac{1}{2}-\epsilon}(0,T;\mathbf{H}^{-\frac{1}{2}-\epsilon}(\Omega))$$

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- Conjecture: $\partial u_t \in L^{\frac{10}{9}}(0,T;\mathbf{L}^{\frac{3}{2}}\Omega)$.

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There is c independent of h such that for $s \in [\frac{3}{10}, \frac{1}{2}]$

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Enough a priori estimates to pass to the limit (+discrete commutator).

FOURIER APPROX IN TORUS

Jopseph Fourier

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- Seek $u_N \in \mathcal{C}^1([0,T];\mathbf{X}_N)$, and $p_N \in C^0(0,T;M_N)$ s.t. $\forall t \in (0,T]$, $\forall v \in \mathbf{X}_N$, and $\forall q \in M_N$

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- Same problem if one uses special bases: e.g. eigenvetors of Stokes operator.
- Bottom line: The commutator property is not strong enough.

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Lemma (Negative result)

$\exists c_1,\ c_2,\ \forall \phi \in W^{1,\infty}(\Omega)$

$$c_1 N^{-1} \|\phi\|_{W^{1,\infty}} \leq \|[P_N, \Phi]\|_{\mathcal{L}(H^1,L^2)} \leq c_2 N^{-1} \|\phi\|_{W^{1,\infty}}.$$

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Lemma (Positive result)

$\exists c$, $\forall \phi \in H^3(\Omega)$, $\forall u_N \in \mathbb{P}_N$

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- $\Rightarrow \|[P_N, \Phi]u_N\|_{L^\infty} \rightarrow 0$ if $u_N \rightarrow u$ in H^1 strong, (weak convergence is not enough!)

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- Does no go to 0, unless $u_N \to u$ in $L^2(0,T;\mathbf{H}^1)$ strong (energy equality).

CONJECTURE

- Result would be true if for all $k_0 > 0$,

$$\lim_{N\to+\infty}\int_0^T \sup_{N<k<N+k_0} (\widehat{u_N\cdot\nabla u_N})_k^2 \,\mathrm{d}t = 0$$

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- Note that if u Leray solution

$$\lim_{k\to+\infty} \int_0^T (\widehat{u \cdot \nabla u})_k^2 \, \mathrm{d}t = 0$$

Consequence of Riemann Lebesgue Lemma, $\|u \cdot \nabla u\|_{L^1(\Omega)}$ is in $L^2(0, T)$.

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