OF NONLINEAR APPROXIMATION OF PDEs IN $L^1(\Omega)$

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Seminar
University of Maryland
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1 Why $L^1$? (Overview from three different fields)
Outline

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2. Approximation in Banach spaces
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5. Steady Hamilton Jacobi (and Burgers) equations in $L^1(\Omega)$
Problem: Given a set of data in $\mathbb{R}^d$, ($d = 1, 2$), construct a spline approximation

\[\text{From J. Lavery, } \textit{Computer Aided Geometric Design,} \ 23 \ (2006) \ 276:296\]
L₁ splines (J. Lavery et al. ARO and NCSU)

**Problem:** Given a set of data in \( \mathbb{R}^d \), \( (d = 1, 2) \), construct a spline approximation.

**Solution:** Minimize the \( \ell^p \)-distance between spline and data.

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Cubic $L^1$ spline

Cubic $L^2$ spline (Least-Squares)$^1$

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$L^1$ splines
$L^1$ for signal processing
$L^1$ for PDEs

$L^1$ splines (J. Lavery et al. ARO and NCSU)

Cubic $L^1$ spline  Cubic $L^2$ spline (Least-Squares)$^3$ (16 × 16)

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$L^1$ splines (J. Lavery et al. ARO and NCSU)

Data ($128 \times 128$)

Cubic $L^1$ spline ($16 \times 16$)

Cubic $L^2$ spline (Least-Squares)

4 Courtesy from J. Lavery et al.
Observations:

- $L^1$ splines are less oscillatory than $L^2$ splines.
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- $L^1$ for PDEs

$L^1$ splines (J. Lavery et al. ARO and NCSU)

Observations:
- $L^1$ splines are less oscillatory than $L^2$ splines.
- $L^1$ splines can compress data better than $L^2$ splines.
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- $L^1$ for PDEs

$L^1$ for image/signal processing/denoising

- Image/signal processing yield **underdetermined** linear systems.
Image/signal processing yield underdetermined linear systems.

Let $F \in \mathbb{R}^{n \times m}$, $\text{rank}(F) = n < m$, and let $y \in \mathbb{R}^n$:

$$\min_{d \in \mathbb{R}^m} \|d\|_{\ell^0} \quad \text{subject to} \quad Fd = y \quad (P_0)$$

where $\|e\|_{\ell^0} = |\{i : e_i \neq 0\}|$, (number of nonzero entries).
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The $\| \cdot \|_{\ell^0}$-norm measure sparsity.
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- We want the sparsest solution.
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The $\| \cdot \|_{\ell^0}$-norm measure sparsity.

We want the sparsest solution.

NP-hard problem in general (Donoho (2004))
Consider the other problem

$$\min_{d \in \mathbb{R}^m} \| d \|_{\ell^1} \quad \text{subject to} \quad Fd = y \quad (P_1)$$
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\((P_1)\) is a convex problem ("Basis pursuit", Chen-Donoho (1999)). 
\((P_1)\) can be solve by linear programming techniques (more tractable than \((P_0)\))
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“For most underdetermined systems of linear equations, the minimal \(\ell^1\)-norm solution is the sparsest solution” (Donoho (2004)), i.e. \((P_0) \Leftrightarrow (P_1)\).
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One could solve the Least-Squares system \(F^T Fd = F^T d\) (Not the sparsest by far. Gives a dense solution).
Let $x_0 \in \mathbb{R}^m$ with $\|x\|_{\ell^0} \leq n$, $n \ll m$. ($x_0$ is a long sparse signal).
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Assume $x_0$ is measured $n$ times: $y_k = \langle x_0, f_k \rangle$, $k \in \{1, \ldots, n\}$.

In matrix form that gives

$$y = Fx_0,$$

where $F \in \mathbb{R}^{n \times m}$. 

\[ \text{Jean-Luc Guermond} \]
\[ \text{NONLINEAR APPROXIMATION OF PDEs in } L^1(\Omega) \]
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We know $y$ and $F$. Can we recover the signal $x_0$?
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We know $y$ and $F$. Can we recover the signal $x_0$?

“If $F$ obeys a uniform uncertainty principle, then $x_0$ is the unique solution to ($P_1$)” (Candes-Tao (2005)). Solving ($P_1$) yields exact recovery of the signal!
Approximation of PDEs in $L^1$. Advection-diffusion

\begin{align*}
  u + \partial_x u - 0.02\nabla^2 u &= 1, \\
  u|_{x=0,x=1} &= 0, \quad \partial_y u|_{y=0,y=1} = 0.
\end{align*}
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Why $L^1$ is doing so well?
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An abstract setting

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- $L \in \mathcal{L}(E; F)$, bounded linear operator, $(L^* : F' \longrightarrow E'$ its adjoint, $E'$, $F'$ duals of $E$, $F$).
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- Assume $L$ is bijective.

Problem: For $f \in F$

\[
\begin{cases}
\text{Find } u \in E \text{ s.t. } \\
Lu = f, \quad \text{in } F.
\end{cases}
\]
Define $J(v) = \|Lv - f\|_F$
A minimization problem

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- Consider the following

$$\begin{align*}
\text{Minimization problem} & \quad \left\{ \begin{array}{l}
\text{Find } u \in E \text{ s.t. } \\
J(u) \leq J(v), \quad \forall v \in E.
\end{array} \right.
\end{align*}$$
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\end{align*}
\]

- Same solution and stability properties as the original pb.
A minimization problem

Proposition

If $F$ is a Hilbert space the solution to the minimization problem is also the unique solution to

$$
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\end{cases}
$$

$\Rightarrow$ Generalization of Least-Squares to Banach spaces.
A minimization problem: discrete setting

\[(E_h)_{h>0}, \text{ finite-dimensional spaces s.t. } E_h \subset E.\]
A minimization problem: discrete setting

- \((E_h)_{h>0}\), finite-dimensional spaces s.t. \(E_h \subset E\).
- Interpolation properties; \(\exists \, W \subset E\) dense normed subspace, \(\exists \, \epsilon(h)_0\), continuous at 0 with \(\epsilon(0) = 0\), s.t.

\[
\forall v \in W, \quad \inf_{v_h \in E_h} \| v - v_h \|_E \lesssim \epsilon(h)\| v \|_W.
\]
A minimization problem: discrete setting

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\]

Discrete problem

\[
\begin{cases}
\text{Find } u_h \in E_h \text{ s.t. } \\
J(u_h) = \min_{v_h \in E_h} J(v_h).
\end{cases}
\]
A minimization problem: discrete setting

Theorem

1. There is at least one global minimizer.
2. There are no local minimizers.
3. All minimizers satisfy the a priori error bound:

\[ \| u - u_h \|_E \lesssim \inf_{v_h \in E_h} \| u - v_h \|_E, \]

and the a posteriori error estimate holds

\[ \| u - u_h \|_E \lesssim \| f - Lu_h \|_F. \]
A minimization problem: discrete setting

Corollary

1. If $u \in W$,
   
   $$\|u - u_h\|_E \lesssim \epsilon(h) \|u\|_W$$

2. If $u \in E$ only
   
   $$\lim_{h \to 0} \|u - u_h\|_E = 0.$$
A minimization problem: discrete setting

Corollary

1. If $u \in W$, 
   \[ \| u - u_h \|_E \lesssim \epsilon(h) \| u \|_W \]

2. If $u \in E$ only 
   \[ \lim_{h \to 0} \| u - u_h \|_E = 0. \]

The \textit{a priori} error estimate is optimal.
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Convergence tests

$\Omega = ]0, 1[^2$ and consider the transport equation

$$\partial_x u = f; \quad u|_{x=0} = u_0,$$

Data

$$f(x, y) = 2\pi \cos(2\pi(x + y)); \quad u_0(y) = \sin(2\pi y),$$

Exact solution:

$$u = \sin(2\pi(x + y)).$$
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Convergence tests

Examples

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$$f(x, y) = 2\pi \cos(2\pi(x + y)); \quad u_0(y) = \sin(2\pi y),$$

Exact solution:

$$u = \sin(2\pi(x + y)).$$

Solve the pb with continuous $P_1$ and $P_2$ F.E. in $L^1(\Omega)$. 
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Convergence tests: $P_1$

$L^1$-norm & $L^2$-norm vs. $h$ 
$L^1(\Omega)$-graph norm & $H^1$-norm
Convergence tests: $P_2$

$L^1$-norm & $L^2$-norm vs. $h$

$L^1(\Omega)$-graph norm & $H^1$-norm

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NONLINEAR APPROXIMATION OF PDEs in $L^1(\Omega)$
Transport equation with shear-layer-like solutions

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NONLINEAR APPROXIMATION OF PDEs in $L^1(\Omega)$
Transport equation with shear-layer-like solutions: Curved transport

$L^1$ sol.  LS sol.
Advection-diffusion equation

\[ u + \partial_x u - 0.00125 \nabla^2 u = 0, \]
\[ u|_{x=0} = 0, \quad u|_{x=1} = 1 \quad \partial_y u|_{y=0, y=1} = 0. \]

2 × 40 × 40 mesh; cell Reynolds number ≈ 40
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Viscosity solutions of PDE's

In $\Omega \in \mathbb{R}^d$ consider:

$$\alpha u + \partial_x u = f; \quad u|_{\partial \Omega} = u_0.$$

Not wellposed in standard sense, but notion of viscosity solution applies (Bardos, Leroux, and Nédélec (1979), Kružkov ...
Viscosity solutions of PDE’s

The viscosity solution can be interpreted as the limit $\epsilon \to 0$ of

$$\alpha u + \partial_x u - \epsilon \nabla^2 u = f; \quad u|_{\partial \Omega} = u_0.$$
Viscosity solutions of PDE's

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One tries to solve the ill-posed pb when $\epsilon / h^2 \ll 1/h$. 
Viscosity solutions of PDE’s

- The viscosity solution can be interpreted as the limit $\epsilon \to 0$ of
  \[
  \alpha u + \partial_x u - \epsilon \nabla^2 u = f; \quad u|_{\partial\Omega} = u_0.
  \]

- One tries to solve the ill-posed pb when $\epsilon/h^2 \ll 1/h$.

A good scheme should be able to approximate the viscosity solution!
Why $L^1$ for viscosity solutions?

Let $u_{\text{visc}}$ be viscosity solution of

$$\nu_{\text{visc}} + \nu'_{\text{visc}} = f, \quad \text{in } (0, 1), \quad \nu_{\text{visc}}(0) = 0, \quad \nu(1) = 0.$$
Why $L^1$ for viscosity solutions?

- Let $u_{\text{visc}}$ be viscosity solution of
  \[ \nu_{\text{visc}} + \nu'_{\text{visc}} = f, \quad \text{in } (0, 1), \quad \nu_{\text{visc}}(0) = 0, \quad \nu(1) = 0. \]

- Let $\tilde{f}$ be the zero extension of $f$ on $(-\infty, +\infty)$.
  Let $\tilde{u}$ solve $\nu + \nu' = \tilde{f} - u_{\text{visc}}(1)\delta(1), \quad \text{in } (-\infty, +\infty), \quad \text{and}$
  \[ \nu(-\infty) = 0, \quad \nu(+\infty) = 0. \]
Why \( L^1 \) for viscosity solutions?

- Let \( u_{\text{visc}} \) be viscosity solution of
  \[
  v_{\text{visc}} + v'_{\text{visc}} = f, \quad \text{in (0, 1)}, \quad v_{\text{visc}}(0) = 0, \quad v(1) = 0.
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- Let \( \tilde{f} \) be the zero extension of \( f \) on \(( -\infty, +\infty)\).
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**Lemma**

\[ \tilde{u}|_{[0,1)} = u_{\text{visc}}. \]
Why $L^1$ for viscosity solutions?

- Let $u_{\text{visc}}$ be viscosity solution of
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  \[ \nu(-\infty) = 0, \quad \nu(+\infty) = 0. \]

**Lemma**

$$\tilde{u}|_{[0,1)} = u_{\text{visc}}.$$  

\(\tilde{u}\) solves a well-posed pb with RHS almost in $L^1(\mathbb{R})$. 
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**LS cannot compute viscosity solutions of PDE’s**

**Lemma**

*Least-Squares method cannot compute viscosity solution.*

⇒ Gives some insight why GaLS has difficulties solving shock and boundary layers pbs.
Viscosity solutions: Theory

Consider the 1D pb:

\[ u + \beta(x)u' = f \text{ in } (0,1), \quad u(0) = 0, \quad u(1) = 0. \]
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- Assume \( 0 < \inf \beta, \sup \beta' < 1 \) and \( f \) is Riemann integrable.
- Use piecewise linear polynomials.
- Use midpoint rule to approximate the integrals.
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**Theorem (Guermond-Popov (2006))**

The best \(L^1\)-approximation converges in \(W^{1,1}_{loc}([0,1])\) to the viscosity solution, and the boundary layer is always located in the last mesh cell.
The idea behind $L^1$

- Let $A$ a $m \times n$ matrix of maximal rank, $m > n$. 
The idea behind $L^1$

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- Let $X$ be a best $\ell^1$-approximation of $AX = F$. 
The idea behind $L^1$

- Let $A$ a $m \times n$ matrix of maximal rank, $m > n$.
- Let $X$ be a best $\ell^1$-approximation of $AX = F$.

**Lemma**

$X$ solves at least $n$ rows in the linear system $AX = F$. 
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The idea: The $\ell^1$-minimizer yields the sparsest residual!
Why $L^1$? (Overview from three different fields)
Approximation in Banach spaces
Numerical illustrations in $L^1(\Omega)$
Viscosity solutions of ill-posed transport equations in $L^1(\Omega)$
Steady Hamilton Jacobi (and Burgers) equations in $L^1(\Omega)$

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**The idea:** The $\ell^1$-minimizer sets $n$ residuals to zero and does not care about the other ones!
Viscosity solutions: numerical experiments

\[ u + \partial_x u = 1; \quad u|_{x=0} = 0, \quad u|_{x=1} = 0. \]
Viscosity solutions: numerical experiments

\[ \partial_x u = 0 \]
\[ u|_{x=0} = 0, \quad u|_{x=1} = 1. \]

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Outline

1. Why $L^1$? (Overview from three different fields)
2. Approximation in Banach spaces
3. Numerical illustrations in $L^1(\Omega)$
4. Viscosity solutions of ill-posed transport equations in $L^1(\Omega)$
5. Steady Hamilton Jacobi (and Burgers) equations in $L^1(\Omega)$
Consider

\[ H(x, u, \nabla u) = 0, \quad u|_{\partial\Omega} = \alpha. \]
Theory

Consider

\[ H(x, u, \nabla u) = 0, \quad u|_{\partial \Omega} = \alpha. \]

- Let \( \Omega \) be a smooth domain in \( \mathbb{R}^2 \).
- \( H \) is convex (\( \|\xi\| \leq c(1 + |H(x, v, \xi)| + |v|) \)).
- \( H \) is Lipschitz.
Consider

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- \( H \) is convex (\( \|\xi\| \leq c(1 + |H(x, v, \xi)| + |v|) \)).
- \( H \) is Lipschitz.
- Assume that \( \Omega, f \) and \( \alpha \) are smooth enough for a viscosity solution to exist \( u \in W^{1,\infty}(\Omega), \nabla u \in BV(\Omega) \) and \( u \)
  p-semi-concave.
  (There is a concave function \( v_c \in W^{1,\infty}(a, b) \) and a function \( w \in W^{2,p}(a, b) \) so that \( u = v_c + w \)).
Discretization

- \{T_h\}_{h > 0} regular mesh family.
- \(X_h^{\alpha h} = \{v_h \in C^0(\omega); \ v_h|K \in \mathbb{P}_1, \forall K \in T_h, \ v_h|_{\partial \Omega} = \alpha_h\}.
- Take \(p > 2\) (\(p > 1\) in one space dimension).
- Define
  \[
  J_h(v_h) = \|H(\cdot, v_h, \nabla v_h)\|_{L^1(\Omega)} + h^{2-p} \sum_{F \in \mathcal{F}_h^i} \int_F (\{-\partial_n v_h\}_+)^p.
  \]
Discretization

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  \]

Compute $u_h \in X_h^{\alpha h}$ s.t., $J(u_h) = \min_{v_h \in X_h^{\alpha h}} J_h(v_h)$.
Discretization

Theorem (Guermond-Popov (2007))

\( u_h \) converges to the viscosity solution strongly in \( W^{1,1}(\Omega) \) (the result holds for arbitrary polynomial degree provided an additional volume entropy is added)
1D convergence tests

\[ \Omega = [0, 1], \quad u + \frac{1}{\pi} (u')^2 = f(x), \quad u(0) = u(1) = -1. \]
1D convergence tests

$$\Omega = [0, 1], \quad u + \frac{1}{\pi}(u')^2 = f(x), \quad u(0) = u(1) = -1.$$ 

Data: $f(x) = -|\cos(\pi x)| + \sin^2(\pi x),$
1D convergence tests

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Data: \( f(x) = -|\cos(\pi x)| + \sin^2(\pi x), \)

Exact solution: \( u(x) = -|\cos(\pi x)|. \)
1D convergence tests

Odd # points (9, 19, 39)  
Even # points (10, 20, 40)
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1D convergence tests

![Graphs showing convergence for odd and even number of points](image)

Jean-Luc Guermond
NONLINEAR APPROXIMATION OF PDEs in $L^1(\Omega)$
1D convergence tests

\[(u')^2 + 3u + \frac{1}{2}x^2 - |x| = 0, \text{ in } (-0.95, 0.95)\]
1D convergence tests

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Boundary condition set so that the viscosity solution \(u_{\text{visc}}\) is
\[u_{\text{visc}}(x) = -\frac{1}{2}x^2 + \frac{2}{3}|x|^{\frac{3}{2}}, \text{ i.e. } u(\pm 0.95) = u_{\text{visc}}(\pm 0.95)\]
1D convergence tests

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\(u_{\text{visc}}\) is in \(W^{1,\infty}(\Omega) \cap W^{2,p}(\Omega)\) for any \(p \in [1, 2)\)
1D convergence tests

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\(u_{\text{visc}}\) is \(p\)-semi-concave for any \(p \in [1, 2)\)
1D convergence tests

-0.95 0 0.95
0.1
0.2

0 0.1 0.2

0.0001 0.001 0.01 0.1 1

$\times 10^{-5}$
$\times 10^{-4}$
$\times 10^{-3}$
$\times 10^{-2}$
$\times 10^{-1}$

Slope 2
$W^{1,1}$-norm
$L^\infty$-norm
$L^1$-norm

Jean-Luc Guermond
NONLINEAR APPROXIMATION OF PDEs in $L^1(\Omega)$
Eikonal equation 2D

\[ \Omega = [0, 1]^2, \quad \| \nabla u \| = 1, \quad u|_{\partial \Omega} = 0. \]
Eikonal equation 2D

\[ \Omega = [0, 1]^2, \quad \| \nabla u \| = 1, \quad u|_{\partial \Omega} = 0. \]

Exact solution: \( u(x) = \text{dist}(x, \partial \Omega) \)
Eikonal equation 2D: $\mathbb{P}_1$ finite elements
Burgers in 1D

\[ \Omega = (0, 1), \quad \alpha u + (u^2/2)' = f, \quad u(0) = -1, \ u(1) = 1. \]
Burgers in 1D

\[ \Omega = (0, 1), \quad \alpha u + (u^2/2)' = f, \quad u(0) = -1, \quad u(1) = 1. \]

Source: \( f(x) = (1 - 2x)(-1 + x - x^2). \)
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**Burgers in 1D: $\mathbb{P}_1$ finite elements**

\[ \alpha = -1 \quad \alpha = 0 \quad \alpha = 1 \]
Fast algorithm?

- The big issues: Minimizing a \textit{non-smooth and non-convex} functional.
Fast algorithm?

- The big issues: Minimizing a non-smooth and non-convex functional.

- Simplex is inefficient.
Fast algorithm?

- The big issues: Minimizing a non-smooth and non-convex functional.
- Simplex is inefficient.
- Idea (in 1D): iterative algorithm based on local $L^1$-minimization. $O(N)$ algorithm for HJ (proved and tested!).
Concluding remarks

- Work in $L^1$ is doable.
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- Working in $L^1$ helps selecting viscosity solutions of some classes of PDE’s.
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- Working in $L^1$ helps selecting viscosity solutions of some classes of PDE’s.

- $L^1$ localize approximation errors (sparsest residual)
Further work

- Extension to time dependent problems.
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- Extension to **time dependent** problems.
- Extension to **systems**.
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- Find fast algorithms to compute best $L^1$-approximations in 2 and three dimensions. (Multigrid version of interior point method?)
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