Entropy viscosity

Jean-Luc Guermond, B. Popov

Department of Mathematics
Texas A&M University
and LIMSI/CNRS, Orsay

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Outline

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The PDE

- Solve the transport equation

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- Deviate as little possible from Galerkin.

Some background

Rk 1 Godonov's theorem: Linear numerical schemes for solving the linear transport equation having the property of not generating new extrema (monotone scheme), can be at most first-order accurate.
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- Slope limiters $\Leftrightarrow$ add nonlinear viscosity.

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$$\left.\begin{array}{r}\text{Need viscosity}\\ \text{Need entropy inequality}\end{array}\right\} \Rightarrow \text{viscosity} \sim \text{entropy residual}.$$

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- Viscosity $\sim$ residual (Hughes-Mallet (1986) Johnson-Szepessy (1990))
- Add entropy to formulation (For Hamilton-Jacobi equations Guermond-Popov (2007))

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- Solution method: Galerkin + entropy viscosity:

$$\int_\Omega (\partial_t u_h + \beta\cdot\nabla u_h) v_h \mathrm{d}\mathbf{x} + \sum_K \int_K \nu_K \nabla u_h \nabla v_h \mathrm{d}\mathbf{x} = 0, \quad \forall v_h$$

The algorithm + time discretization

- Use an explicit time stepping: BDF, RK3, RK4, etc.

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- Idea: make the viscosity explicit.

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$$\|D_h\|_{\infty,K} := \|\frac{3(u_h^n)^2 - 4(u_h^{n-1})^2 + (u_h^{n-2})^2}{2\Delta t} + \beta\cdot\nabla(u_h^n)^2\|_{\infty,K}$$

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- Compute u_h^{n+1}

$$\frac{3}{2\Delta t}\int_\Omega u_h^{n+1} v_h \mathrm{d}\mathbf{x} = \int_\Omega \frac{1}{2\Delta t}(4u_h^n - u_h^{n-1}) - \beta\cdot\nabla(2u_h^n - u_h^{n-1}) v_h \mathrm{d}\mathbf{x}$$
$$- \sum_K \int_K \nu_K \nabla(2u_h^n - u_h^{n-1})\nabla v_h \mathrm{d}\mathbf{x}, \quad \forall v_h$$

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Let u_h be the finite element approximation. If u in $H^1(\Omega)$ then
$\|u - u_h\|_{L^2} \leq c\, h^{\frac{1}{2}} \|u\|_{H^1}$ *if D_h based on residual.*
$\|u - u_h\|_{L^2} \leq c\, h^{\frac{1}{4}} \|u\|_{H^1}$ *if D_h based on quadratic entropy.*

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Theorem

In one space dimension with $\mathbb{P}_1$ finite element and residual viscosity,

$$\|u_h\|_{L^\infty} \leq (1 + c h^{\frac{1}{2}} \log(1/h)) \|f\|_{L^\infty}.$$

Numerical tests, smooth case, time-dependent transport

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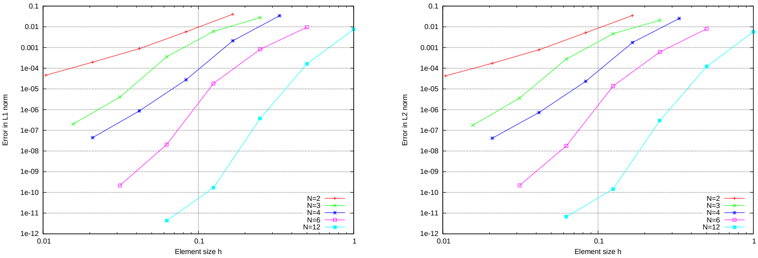
h	$\mathbb{P}_1$ Stab.				$\mathbb{P}_1$ Gal.			
	L^2	rate	L^1	rate	L^2	rate	L^1	rate
2.00E-1	2.5893E-1	-	3.6139E-1	-	1.7055E-1	-	2.7910E-1	-
1.00E-1	9.7934E-2	1.403	1.3208E-1	1.452	7.2792E-2	1.228	1.1463E-1	1.284
5.00E-2	1.9619E-3	2.320	2.7310E-3	2.274	1.0993E-2	2.727	1.8321E-2	2.645
2.50E-2	3.5360E-4	2.472	5.1335E-3	2.411	2.0080E-3	2.453	3.4351E-3	2.415
1.25E-2	6.4959E-4	2.445	1.0061E-3	2.351	4.6904E-4	2.098	7.9146E-4	2.118
1.00E-2	3.9226E-4	2.261	6.3555E-4	2.058	3.0943E-4	1.864	5.3982E-4	1.715
6.25E-3	1.4042E-4	2.186	2.3829E-4	2.087	1.2295E-4	1.964	2.1743E-4	1.935

Table: $\mathbb{P}_1$ approximation.

Numerical tests, smooth case, (spectral elements)



Linear transport problem with smooth initial condition. Errors in L^1 (at left) and L^2 (at right) norms vs h for different polynomial approximation degree N.

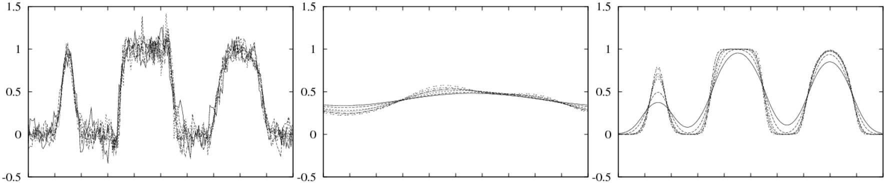
1D Numerical tests, BV solution

- linear transport

$$\partial_t u + \partial_x u = 0, \quad u_0(x) = \begin{cases} e^{-300(2x-0.3)^2} & \text{if } |2x-0.3| \le 0.25, \\ 1 & \text{if } |2x-0.9| \le 0.2, \\ \left(1 - \left(\frac{2x-1.6}{0.2}\right)^2\right)^{\frac{1}{2}} & \text{if } |2x-1.6| \le 0.2, \\ 0 & \text{otherwise.} \end{cases}$$

- Periodic boundary conditions.
- Spectral elements in 1D on random meshes.
- Long time integration, 100 periods.

1D Numerical tests, BV solution



Long time integration, $t = 100$, for polynomial degrees $k = 2, \ldots 8$, #d.o.f.=200. Galerkin (left); Constant viscosity (center); Entropy viscosity (right).

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5.00E-2	5.5707E-2	0.394	1.3704E-2	0.795	5.7999E-2	0.424	3.4561E-2	0.538
2.50E-2	4.2522E-2	0.389	8.0365E-3	0.770	4.5870E-2	0.338	2.5407E-2	0.444
1.25E-2	3.2409E-2	0.392	4.6749E-3	0.782	3.6951E-2	0.312	1.8913E-2	0.426
1.00E-2	2.9812E-2	0.374	3.9421E-3	0.764	3.4146E-2	0.354	1.7187E-2	0.429
6.25E-3	2.4771E-2	0.394	2.7200E-3	0.790	2.8974E-2	0.350	1.4406E-2	0.376

Table: $\mathbb{P}_2$ approximation.

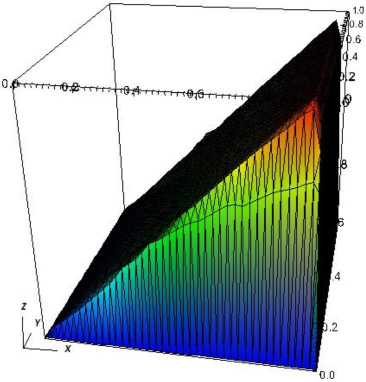
2D BV solution

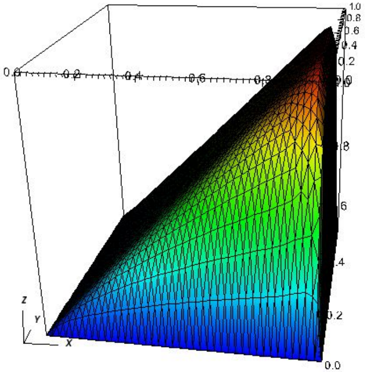
- On $\Omega = (0,1)^2$, solve

$$\partial_t u + \partial_x u - \epsilon \nabla^2 u = 1, \quad u_\Gamma = 0, \quad u(t=0) = 1.$$

- $\epsilon = 10^{-5}$.
- Use $\mathbb{P}_2$ finite elements + RK4, Delaunay mesh, $h \approx 1/40$.

2D BV solution, Entropy vs. Viscous





Nonlinear scalar conservation laws

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$$\partial_t E(u) + \partial_x F(u) + \partial_y G(u) \leq 0$$

for all entropy pair $E(u)$, $F(u) = \int E'(u) f'(u) \mathrm{d}u$, $G(u) = \int E'(u) g'(u) \mathrm{d}u$

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- Compute solution in $(0,1)^2$ at $t = \frac{1}{2}$.

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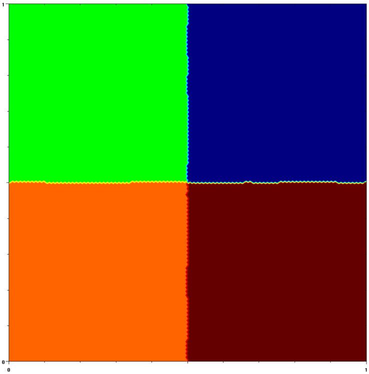
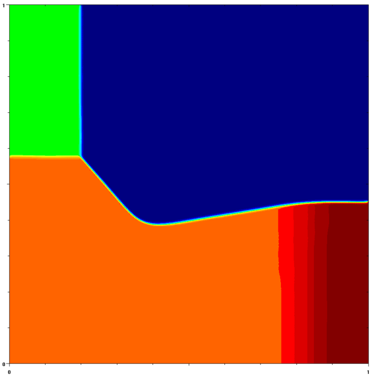




Initial data

$\mathbb{P}_1$ FE, $3\,10^4$ nodes

Buckley Leverett, $\mathbb{P}_2$ FE

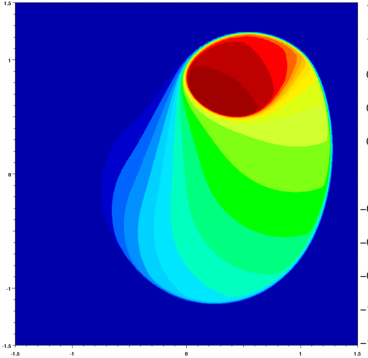
- Solve $\partial_t u + \partial_x f(u) + \partial_y g(u) = 0$.

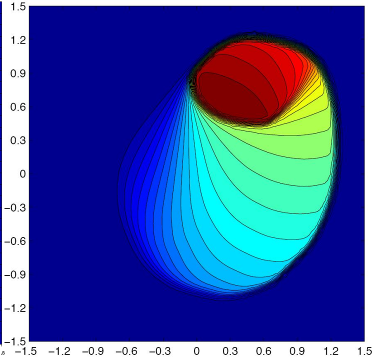
$$f(u) = \frac{u^2}{u^2+(1-u)^2}, \qquad g(u) = f(u)(1-5(1-u)^2)$$

Non-convex fluxes (composite waves)

$$u(x,y,0) = \begin{cases} 1, & \sqrt{x^2+y^2} \leq 0.5 \\ 0, & \text{else} \end{cases}$$

Buckley Leverett, $\mathbb{P}_2$ FE





KPP (WENO + superbee limiter fails), $\mathbb{P}_2$ FE

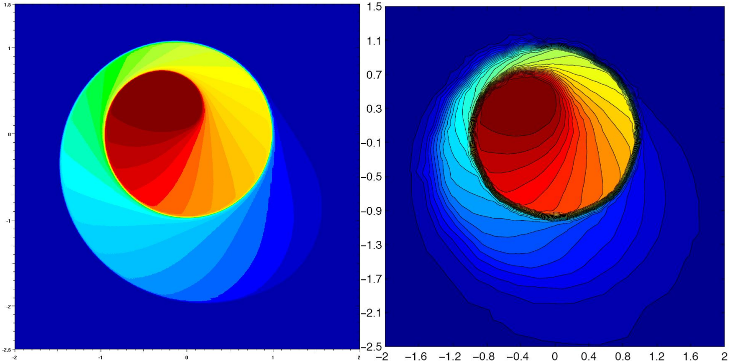
- Solve $\partial_t u + \partial_x f(u) + \partial_y g(u) = 0$.

$$f(u) = \sin(u), \qquad g(u) = \cos(u)$$

Non-convex fluxes (composite waves)

$$u(x,y,0) = \begin{cases} \frac{7}{2}\pi, & \sqrt{x^2+y^2} \leq 1 \\ \frac{1}{4}\pi, & \text{else} \end{cases}$$

KPP (WENO + superbee limiter fails), $\mathbb{P}_2$ FE



Euler flows

- Solve compressible Euler equations

$$\partial_t \rho + \nabla\cdot(\rho \mathbf{u}) = 0$$
$$\partial_t(\rho \mathbf{u}) + \nabla\cdot(\rho \mathbf{u} \otimes \mathbf{u} + p\mathbb{I}) = 0$$
$$\partial_t(E) + \nabla\cdot(\mathbf{u}(E + p)) = 0$$
$$\rho e = E - \frac{1}{2}\rho \mathbf{u}^2, \quad T = (\gamma - 1)e \quad T = \frac{p}{\rho}$$

Initial data + BCs

- Use continuous finite elements of degree p.

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Initial data + BCs

- Use continuous finite elements of degree p.
- Deviate as little possible from Galerkin.

The algorithm

- Compute the entropy $S_h = \frac{\rho_h}{\gamma-1}\log(p_h/\rho_h^\gamma)$

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- Solution method: Galerkin + entropy viscosity + thermal conductivity

1D burgers + Fourier

- Solution method: Fourier + BDF4 + entropy viscosity

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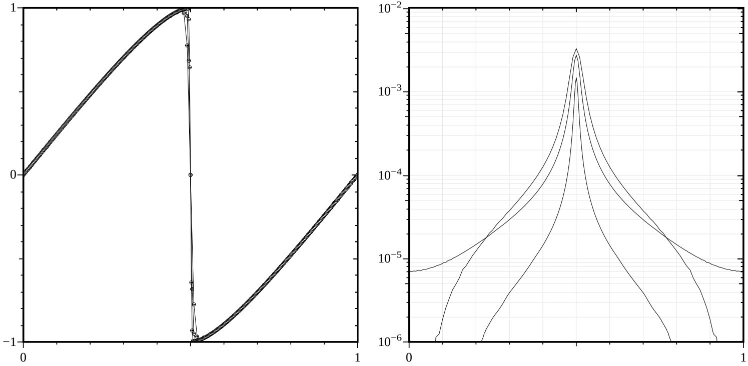

Figure: Left: u_N and right: $\nu_N(u_N)$, for Burgers at $t = 0.25$ with $N = 50$, 100, and 200.

1D Nonconvex flux + Fourier

- Consider $\partial_t + \partial_x f(u) = 0$, $u(x,0) = u_0(x)$

$$f(u) = \begin{cases} \frac{1}{4}u(1-u) & \text{if } u < \frac{1}{2}, \\ \frac{1}{2}u(u-1) + \frac{3}{16} & \text{if } u \geq \frac{1}{2}, \end{cases} \qquad u_0(x) = \begin{cases} 0, & x \in (0, 0.25], \\ 1, & x \in (0.25, 1] \end{cases}$$

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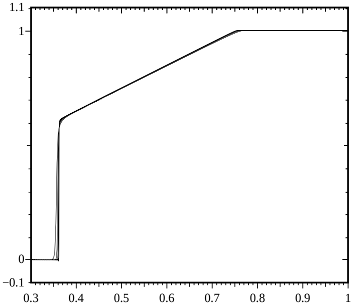

Non-convex flux problem u_N at $t = 1$ with $N = 200$, 400, 800, and 1600.

1D Euler flows + Fourier

- Solution method: Fourier + BDF4 + entropy viscosity

1D Euler flows + Fourier

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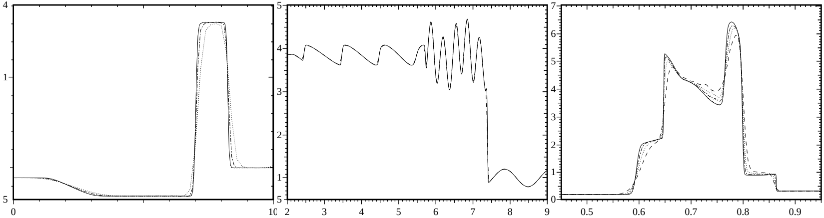


Figure: Lax shock tube, $t = 1.3$, 50, 100, 200 points. Shu-Osher shock tube, $t = 1.8$, 400, 800 points. Right: Woodward-Collela blast wave, $t = 0.038$, 200, 400, 800, 1600 points.

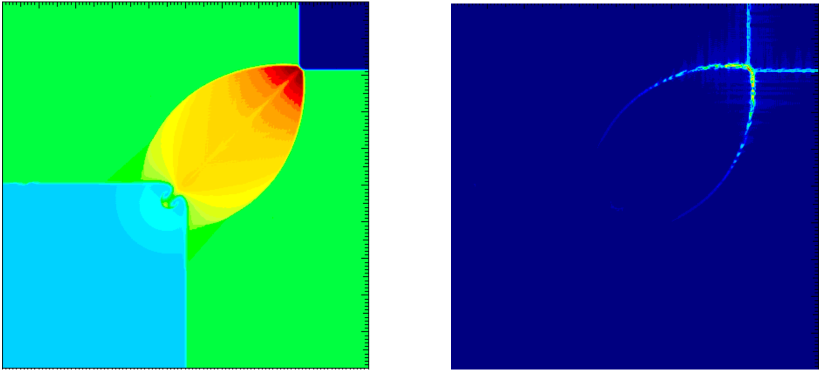
2D Euler flows + Fourier

- Domain $\Omega = (-1, 1)^2$
- Rieman problem with the initial condition:

$$
\begin{aligned}
&0 < x < 0.5 \text{ and } 0 < y < 0.5, && p = 1, \rho = 0.8, \mathbf{u} = (0, 0),\\
&0 < x < 0.5 \text{ and } 0.5 < y < 1, && p = 1, \rho = 1, \mathbf{u} = (0.7276, 0),\\
&0.5 < x < 1 \text{ and } 0 < y < 0.5, && p = 1, \rho = 1, \mathbf{u} = (0, 0.7276),\\
&0 < x < 0.5 \text{ and } 0.5 < y < 1, && p = 0.4, \rho = 0.5313, \mathbf{u} = (0, 0).
\end{aligned}
$$

- Solution at time $t = 0.2$.

2D Euler flows + Fourier



Euler benchmark, Fourier approximation: Density (at left), $0.528 < \rho_N < 1.707$ and viscosity (at right), $0 < \mu_N < 3.410^{-3}$, at $t = 0.2$, $N = 400$.

Mach 3 Wind Tunnel with a Step, $\mathbb{P}_1$ finite elements

- Mach 3 Wind Tunnel with a Step (Standard Benchmark since Woodward and Colella (1984))

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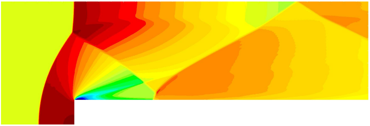
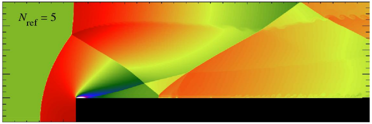


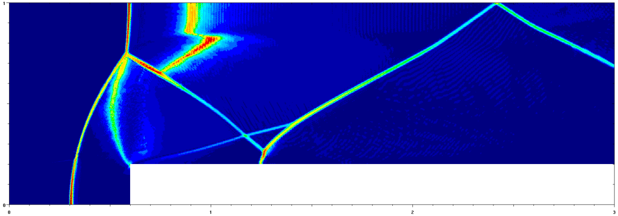
$\mathbb{P}_1$ FE, $1.3\,10^5$ nodes
Log(density)



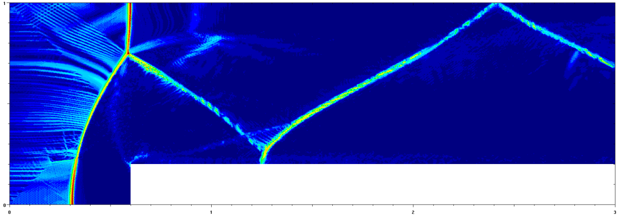
Flash Code, adaptive *PPM*, $\sim 4.9\,10^6$ nodes

movie, www.math.duke.edu/~ying

Mach 3 Wind Tunnel with a Step



Viscosity for mass



Viscosity for momentum

Mach 10 Double Mach reflection

- Right-moving Mach 10 shock makes 60^o angle with x-axis (Standard Benchmark, Woodward and Colella (1984))

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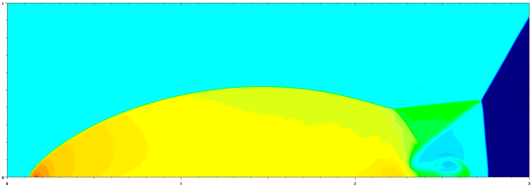
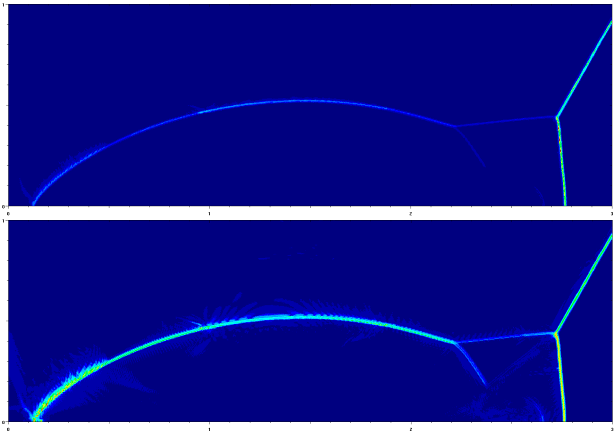


$\mathbb{P}_1$ FE, $1.5\,10^5$ nodes, $t = 0.2$

movie, mngrid.ucsd.edu/˜akritsuk

Mach 10 Double Mach reflection



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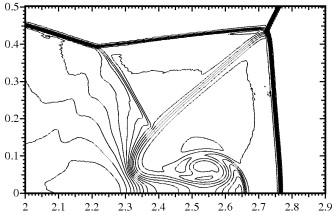
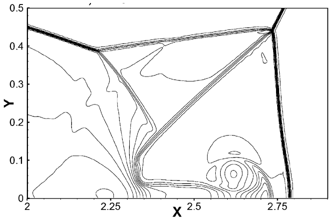


$\mathbb{P}_1$ FE, $1.5\,10^5$ nodes



WENO5, $\sim 1.7\,10^5$ nodes