Discontinuous Galerkin Methods for Anisotropic and Locally Vanishing Diffusion with Advection

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Let $\Omega \subset \mathbb{R}^d$ bounded, open and connected Lipschitz domain.

Let $P_\Omega \overset{\text{def}}{=} \{ \Omega_i \}_{i=1}^N$ a partition of $\Omega$ into Lipschitz connected subdomains.

We consider the following problem:

$$
\nabla \cdot (-\nu \nabla u + \beta u) + \mu u = f,
$$

where

- $\nu \in [L^\infty(\Omega)]^{d, d} \geq 0$ symmetric piecewise constant on $P_\Omega$ s.t.
  $$
  \|\nu\|_{L^\infty(\Omega)}^{d,d} \leq 1;
  $$

- $\beta \in [C^0, 1(\overline{\Omega})]^d$ is s.t. $\nabla \cdot \beta \in L^\infty(\Omega)$;

- $\mu \in L^\infty(\Omega)$ is s.t. $\mu + \frac{1}{2} \nabla \cdot \beta \geq \mu_0$ with $\mu_0 > 0$. 

A One-Dimensional Example

\[
\begin{aligned}
\begin{cases}
  (-\nu u_\varepsilon' + u_\varepsilon)' = 0, & \text{in } (0, 1), \\
  u_\varepsilon(0) = 1, \\
  u_\varepsilon(1) = 0.
\end{cases}
\end{aligned}
\]

\[\lim_{\varepsilon \to 0} u_\varepsilon = u, \text{ with } u|_{\Omega_1 \cup \Omega_2} = 1, u|_{\Omega_3} = 3(x - 1), \text{ discontinuous at } x = \frac{2}{3}\]
Let
\[ \Gamma \overset{\text{def}}{=} \{ x \in \Omega; \exists i_1, i_2 \in \{1, \ldots, N_\Omega\}, x \in \partial \Omega_{i_1} \cap \partial \Omega_{i_2} \}, \]
where \( i_1 \) and \( i_2 \) are s.t. \( (n^\top \nu n)|_{\Omega_{i_1}} \geq (n^\top \nu n)|_{\Omega_{i_2}} \).

We define the elliptic-hyperbolic interface as
\[ I \overset{\text{def}}{=} \{ x \in \Gamma; (n^\top \nu n)(x)|_{\Omega_{i_1}} > 0, (n^\top \nu n)(x)|_{\Omega_{i_2}} = 0 \}, \]

The following relevant subsets of \( I \) can be identified:
\[ I^+ \overset{\text{def}}{=} \{ x \in \Gamma; (n^\top \nu n)(x)|_{\Omega_{i_1}} > 0, (n^\top \nu n)(x)|_{\Omega_{i_2}} = 0 \text{ and } \beta \cdot n_1 > 0 \}, \]
\[ I^- \overset{\text{def}}{=} \{ x \in \Gamma; (n^\top \nu n)(x)|_{\Omega_{i_1}} > 0, (n^\top \nu n)(x)|_{\Omega_{i_2}} = 0 \text{ and } \beta \cdot n_1 < 0 \}. \]
For all scalar $\varphi$ with a (possibly two-valued) trace on $\Gamma$, define
\[
\{\varphi\} \overset{\text{def}}{=} \frac{1}{2} (\varphi|_{\Omega_i_1} + \varphi|_{\Omega_i_2}), \quad [\varphi] \overset{\text{def}}{=} \varphi|_{\Omega_i_1} - \varphi|_{\Omega_i_2}.
\]

We require that
\[
[\!u\!] = 0 \text{ on } I^+.
\]
Continuity is not enforced on $I^-$, where the solution may be discontinuous.
When $\nu$ is isotropic the above conditions coincide with those derived in [Gastaldi and Quarteroni, 1989] for the one-dimensional case.
A Two-Dimensional Exact Solution I

For a suitable rhs,

\[ u = \begin{cases} 
(\theta - \pi)^2, & \text{if } 0 \leq \theta \leq \pi, \\
3\pi(\theta - \pi), & \text{if } \pi < \theta < 2\pi.
\end{cases} \]
A Two-Dimensional Exact Solution II
Let

\[ \kappa \overset{\text{def}}{=} \nu^{1/2}. \]

The model problem is equivalent to

\[
\begin{cases}
\sigma + \kappa \nabla u = 0, & \text{in } \Omega \setminus I, \\
\nabla \cdot (\kappa \sigma + \beta u) + \mu u = 0, & \text{in } \Omega.
\end{cases}
\]

\( \sigma \) is only defined in \( \Omega \setminus I \) since \( u \) can be discontinuous across \( I^- \), so that \( \kappa \nabla u \) has no meaning.

For brevity of notation, we let \( y = (y^\sigma, y^u) \) and introduce the symbol

\[ \Phi(y) \overset{\text{def}}{=} \kappa y^\sigma + \beta y^u. \]
The graph space of the associated operator is

\[ W \overset{\text{def}}{=} \{ y \in L; \ k \nabla y^u \in L_\sigma \text{ and } \nabla \cdot \Phi(y) \in L_u \}, \]

where we have set

\[ L_\sigma \overset{\text{def}}{=} [L^2(\Omega \setminus I)]^d, \quad L_u \overset{\text{def}}{=} L^2(\Omega), \quad L \overset{\text{def}}{=} L_\sigma \times L_u. \]

The space choice together with \((\text{cont. } u)\) gives

\[ \{ \Phi(z) \cdot n \} = 0, \quad \text{on } \Gamma, \]

\[ \llbracket z^u \rrbracket = 0, \quad \text{on } \Gamma \setminus I^- \.]
Use the framework of symmetric Friedrichs systems developed in [Ern et al., 2006, Ern and Guermond, 2006]

Define the following operators:

\[
\mathcal{L}(L; L) \ni K : z \mapsto (z^\sigma, \mu z^u),
\]

\[
\mathcal{L}(W; L) \ni A : z \mapsto (\kappa \nabla z^u, \nabla \cdot \Phi(z)).
\]

Equip the space \( W \) with the following norm:

\[
\|y\|^2_W \overset{\text{def}}{=} \|y\|^2_L + \|Ay\|^2_L.
\]
We introduce the operators $M$ and $D$ s.t., for all $z, y \in W \times W$,

$$\langle Dz, y \rangle_{W', W} = \int_{\partial \Omega} y^t D z, \quad \langle Mz, y \rangle_{W', W} = \int_{\partial \Omega} y^t M z,$$

with

$$D = \begin{bmatrix} 0 & \kappa n_F \newline (\kappa n_F)^t & \beta \cdot n \end{bmatrix}, \quad M = \begin{bmatrix} 0 & -\alpha \kappa n_F \\ \alpha (\kappa n_F)^t & |\beta \cdot n| \end{bmatrix}.$$

$$\langle Dz, y \rangle_{W', W} \overset{\text{def}}{=} (Az, y)_L - (z, \tilde{A}y)_L$$

Let $V = \text{Ker}(M - D) \subset W$. We can prove that

- If $\alpha = 1$, $V = \{ w \in W; \ w^u|_{\{ x \in \partial \Omega; \ n^t \kappa n \neq 0 \text{ or } \beta \cdot n < 0 \}} = 0 \}.$
- If $\alpha = -1$, $V = \{ w \in W; \ \Phi(w) \cdot n = \frac{1}{2} (\beta \cdot n + |\beta \cdot n|) w^u \}.$
Consider the bilinear form

\[ a(z, y) \overset{\text{def}}{=} ((K + A)z, y)_L + \int_{l^+} (\beta \cdot n_1)[z^u] [y^u] + \frac{1}{2} \langle (M - D)(z), y \rangle_{W',W} \]

and the associated problem

\[
\begin{cases}
\text{Find } z \in W \text{ such that, for all } y \in W, \\
a(z, y) = (f, y^u)_{L^u}.
\end{cases}
\]

(weak)

Then

- if \( z \) solves (weak), then it is also a solution of problem (mixed) with boundary conditions \((M - D)(z)|_{\partial \Omega} = 0;\)
- \([z^u] = 0 \text{ on } l^+.\)
Discontinuous Galerkin methods rely on a piecewise fully discontinuous approximation.

To some extent, they can be seen as an extension of finite volume methods.

Their analysis can be performed exploiting many classical results valid for continuous Galerkin approximations.
The bilinear form $b_h$ associated to a DG method for a linear PDE problem can be written as

$$b_h(u, v) = b_h^V(u, v) + b_h^i(u, v) + b_h^\partial(u, v),$$

where

- $b_h^V$ correspond to the standard Galerkin terms;
- $b_h^i$ contains interface terms intended,
  - (i) to penalize the non-conforming component of the discrete solution,
  - (ii) to ensure the consistency of the method;
- $b_h^\partial$ collects boundary terms used to weakly enforce boundary conditions.
Why DG Methods?

**Pros**
- Discontinuous solutions are naturally handled so long as the discontinuities are aligned with the mesh (!).
- Convergence estimates depend only on local Sobolev regularity inside each element (high-order even for solution that display poor global regularity) (!).
- There is great freedom in the choice of bases and of element shapes.
- $hp$-adaptivity can be easily implemented.
- Boundary conditions are weakly enforced (no problems with corner nodes).
- Non-matching grids allowed.

**Cons**
- High(er) computational cost.
Let \( \{ T_h \}_{h>0} \) be a family of affine meshes of \( \Omega \) compatible with \( P_\Omega \).

- \( \mathcal{F}_h^i \) will denote the set of interfaces, \( \mathcal{F}_h^\partial \) the set of boundary faces and \( \mathcal{F}_h \) \( \stackrel{\text{def}}{=} \mathcal{F}_h^i \cup \mathcal{F}_h^\partial \).

- We define the following discontinuous finite element space on \( T_h \):
  \[
P_{h,p} \overset{\text{def}}{=} \{ v_h \in L^2(\Omega); \ \forall T \in T_h, \ v_h|_T \in \mathbb{P}_p(T) \}.
  \]

- Furthermore, we assume that mesh regularity and usual inverse and trace inequalities hold.
For all $F^i_h \ni F = \partial T_1 \cap \partial T_2$ we define

$$\lambda_i \overset{\text{def}}{=} \sqrt{n^t \nu n}|_{T_i}, \quad i \in \{1, 2\},$$

and, without loss of generality, we assume that $\lambda_1 \geq \lambda_2$.

Similarly, for $F \in \mathcal{F}_h^\partial$,

$$\lambda \overset{\text{def}}{=} \sqrt{n^t \nu n}.$$

Observe that the discrete counterpart of $I^\pm$ is not identified.
For all $F \in \mathcal{F}_h$, let

$$[L^2(F)]^2 \ni \omega = (\omega_1, \omega_2), \quad \omega_1 + \omega_2 = 1 \text{ for a.e. } x \in F.$$  

For all $F \ni F = \partial T_1 \cap \partial T_2$ and for a.e. $x \in F$ we define the following weighted trace operators:

$$\{\varphi\}_{\omega} \overset{\text{def}}{=} \omega_1 \varphi|_{T_1} + \omega_2 \varphi|_{T_2}, \quad [\varphi]_{\omega} \overset{\text{def}}{=} 2(\omega_2 \varphi|_{T_1} - \omega_1 \varphi|_{T_2}).$$

When $\omega = (\frac{1}{2}, \frac{1}{2})$, the usual average and jump operators are recovered and subscripts are omitted.
(C1) The bilinear form $a_h$ is $L$-coercive and strongly consistent.
Design Constraints

(C1) The bilinear form $a_h$ is $L$-coercive and strongly consistent.

(C2) The elliptic-hyperbolic interfaces are not identified \textit{a priori}, but an automatic detection mechanism is devised instead.
(C1) The bilinear form $a_h$ is \textit{L-coercive} and \textit{strongly consistent}.

(C2) The elliptic-hyperbolic interfaces are not identified \textit{a priori}, but an automatic detection mechanism is devised instead.

(C3) Suitable \textit{stabilizing terms} are incorporated to control the fluxes.
In what follows the symbols \( \lesssim \) and \( \gtrsim \) will be used for inequalities that hold up to a real positive constant
- independent of the meshsize \( h \),
- independent of the diffusivity \( \nu \),
- possibly depending on the mesh regularity, the polynomial degree, \( \| \beta \|_{L^\infty(\Omega)}^d \) and \( \| \mu \|_{L^\infty(\Omega)} \).

\( S_F \) and \( M_F \) will denote two operators s.t.

\[
\forall F \in \mathcal{F}_h^i, \quad S_F \geq 0,
\]

\[
\forall F \in \mathcal{F}_h^\partial, \quad M_F = \begin{pmatrix} 0 & -\alpha \kappa n_F \\ \alpha (\kappa n_F)^t & M_{F}^{uu} \end{pmatrix} \quad \text{and} \quad M_{F}^{uu} \geq 0.
\]

For the associated seminorms we shall use the symbols \( | \cdot |_M \) and \( | \cdot |_J \) respectively.
Consider the following discrete bilinear form satisfying (C2):

\[ a_h(z, y) \overset{\text{def}}{=} \sum_{T \in T_h} \left[ (Kz, y)_L, \tau + (Az, y)_L, \tau \right] - 2 \sum_{F \in \mathcal{F}_h^i} \left( \{ \Phi(z) \cdot n \}, \{ y^u \} \right)_L, F + \left( \|z^u\|, \frac{1}{4} \|\Phi(y) \cdot n\| \right)_L, F - \frac{\beta \cdot n_1}{2} \{ y^u \} )_{L, F} \]

\[ + \frac{1}{2} \sum_{F \in \mathcal{F}_h^\partial} ( (M_F - D)z, y)_L, F + \sum_{F \in \mathcal{F}_h^i} (S_F(\|z^u\|), \|y^u\|)_L, F. \]

The associated discrete problem reads

\[ \left\{ \begin{array}{l}
\text{Seek } z_h \in W_h \text{ such that } \\
a_h(z_h, y_h) = (f, y_h^u)_L, F, \quad \forall y_h \in W_h.
\end{array} \right. \]
**L-coercivity and Consistency**

**Lemma**

Let \( z \) and \( z_h \) solve (weak) and (discrete) respectively and suppose that

\[\forall F \in \mathcal{F}_h, \quad \omega = \begin{cases} \left( \frac{\lambda_1}{\lambda_1+\lambda_2}, \frac{\lambda_2}{\lambda_1+\lambda_2} \right), & \text{if } \lambda_1 > 0, \\ \left( \frac{1}{2}, \frac{1}{2} \right), & \text{otherwise} \end{cases},\]

(\text{choice } \omega)

\[M_F^{uu} \overset{\text{def}}{=} \frac{|\beta \cdot n|}{2} + \frac{\lambda^2}{h_F}, \quad S_F \overset{\text{def}}{=} \frac{|\beta \cdot n|}{2} + \frac{\lambda_2^2}{h_F},\]

(LDG)

where by definition, \( \lambda_2 = \min(\lambda_1, \lambda_2) \). Then

(i) \( a_h \) is \textit{L-coercive}, i.e., for all \( h \) and for all \( y \) in \( W(h) \),

\[a_h(y, y) \gtrsim \|y\|^2_L + |y^u|^2_J + |y^u|^2_M.\]

(ii) \( a_h \) is \textit{strongly consistent}, i.e., for all \( h > 0 \),

\[\forall y_h \in W_h, \quad a_h(z - z_h, y_h) = 0.\]
Lemma (Stability)

Equip the space $W(h)$ with the following discrete norm:

$$\| y \|_{h,B}^2 \overset{\text{def}}{=} \| y \|_L^2 + \| y^u \|_J^2 + \| y^u \|_M^2 + \sum_{T \in \mathcal{T}_h} \| \kappa \nabla y^u \|_{L^\sigma,T}^2.$$ 

The following bound holds:

$$\forall z_h \in W_h, \quad \| z_h \|_{h,B} \lesssim \sup_{y_h \in W_h \setminus \{0\}} \frac{a_h(z_h, y_h)}{\| y_h \|_{h,B}}.$$
Theorem (Convergence)

Assume that
(i) $W_h = [P_{h,p_{\sigma}}]^d \times P_{h,p_u}$ with $p_u - 1 \leq p_{\sigma}$;
(ii) $z \in [H^{p_{\sigma}+1}(I_h)]^d \times H^{p_u+1}(I_h)$ solve (weak) and $z_h$ solve (discrete);
(iii) (choice $\omega$) and (LDG) are satisfied.

Then,
$$
\|Z - z_h\|_{h,B} \lesssim h^{p_u} \|Z\|_{[H^{p_{\sigma}+1}(I_h)]^d \times H^{p_u+1}(I_h)}.
$$

Remark

The above estimate can be improved using a duality argument when $\nu > 0$. 
The $\sigma$-component of the unknown can be locally eliminated, thus reducing the complexity of the problem.

In what follows we shall discuss a variant with smaller local problem size and reduced matrix sparseness.

∀ $F \in \mathcal{F}_h$, ∀ $\varphi \in L^2(F)$ define the lifting operator $r_{F,\kappa}$ as

$$\forall \tau_h \in \Sigma_h, \quad (r_{F,\kappa}(\varphi), \tau_h)_{L_\sigma} \overset{\text{def}}{=} \begin{cases} 
(\varphi n, \kappa \tau_h)_{L_\sigma,F}, & \text{if } F \in \mathcal{F}_h^0, \\
(\varphi n_1, \{\kappa \tau_h\} \omega)_{L_\sigma,F}, & \text{if } F \in \mathcal{F}_h^i,
\end{cases}$$

where $\omega \overset{\text{def}}{=} (1, 1) - \omega$.

Moreover, we let

$$R_{\kappa}(\varphi) \overset{\text{def}}{=} \sum_{F \in \mathcal{F}_h} r_{F,\kappa}(\varphi).$$
Proceeding in a similar way as in [Arnold et al., 2002], it is possible to prove that, for all \((\sigma, u) \in \mathcal{W}(h)\) and for all \((0, v) \in \mathcal{W}(h)\),

\[
a_h((\sigma, u), (0, v)) = \sum_{T \in \mathcal{T}_h} \left[ (\kappa \nabla u - R_\kappa([u]), \kappa \nabla v - R_\kappa([v]))_{L_u,T} + (\mu u, v)_{L_u,T} \right] \\
- \sum_{T \in \mathcal{T}_h} (u, \beta \cdot \nabla v)_{L_u,T} + \sum_{F \in \mathcal{F}_h^\partial} (M^{uu}_F(u) + (\beta \cdot n) u, v)_{L_u,F} \\
+ \sum_{F \in \mathcal{F}_h^i} ((\beta \cdot n_1)\{u\}, [v])_{L_u,F} + \sum_{F \in \mathcal{F}_h^i} (S_F(u), v)_{L_u,F}.
\]

Observe that \(\sigma\) does not appear in the expression in the RHS, i.e., we have found a decoupled problem for the sole primal unknown.
The stencil of the local problem for the elimination of the $\sigma$-component of the unknown is depicted below:

We can prove that the only term involving the dashed elements is

$$
\sum_{T \in \mathcal{T}_h} (R_{\kappa}([u]), R_{\kappa}([v]))_{L_{uT}} \overset{\text{def}}{=} -\rho_h(u, v).
$$

We are then prompted to consider

$$
a_h^p(z, y) \overset{\text{def}}{=} a_h(z, y) + \rho_h(z^u, y^u).
$$
The corresponding method displays the same properties as the original one provided we take

\[ M_F^{uu} \equiv \frac{|\beta \cdot n|}{2} + \eta \frac{\lambda^2}{h_F}, \quad S_F \equiv \frac{|\beta \cdot n|}{2} + \eta \frac{\lambda^2}{h_F}, \]

with \( \eta > 0 \) large enough.

The resulting method has formal analogies with the IP method first proposed in [Baker, 1977].

*The reason why \( \eta \) must be large enough is that the term \( \rho h \) must be overcome by the stabilization terms in order to recover L-coercivity.*
### Convergence Results

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<th>$P_{h,1}$</th>
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</table>

Jean-Luc Guermond  
DG Methods for Anisotropic and Locally Vanishing Diffusion
An Example With Strongly Anisotropic Diffusivity

\[
\nu_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \quad \beta = (-5, 0)
\]

\[
\nu_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0.25 \end{bmatrix}
\]
An Example With Strongly Anisotropic Diffusivity I

(a) $U_h = P_{h,1}$

(b) $U_h = P_{h,2}$
An Example With Strongly Anisotropic Diffusivity II

(c) $U_h = P_{h,3}$

(d) $U_h = P_{h,4}$
Elliptic-hyperbolic interface conditions were generalized to the tensor diffusivity case, and the normal diffusion component was shown to play a key role.

The notion of **elliptic and hyperbolic subdomains** is somewhat misleading, since the diffusion may possibly vanish along only one direction.

Consistency and stabilization terms have been carefully chosen in order to ensure consistency of the method at the elliptic-hyperbolic interface.

When suitable **weighted trace operators** are used in consistency terms, the elliptic-hyperbolic interface is **automatically detected**, and no further intervention is needed.

The method admits cheaper variants with reduced stencil and alternative penalization forms can be devised that keeps arbitrariness in choosing the parameters to a minimum.
Unified analysis of discontinuous Galerkin methods for elliptic problems.

Finite element methods for elliptic equations using nonconforming elements.

Discontinuous Galerkin methods for Friedrichs systems. I. General theory.

An intrinsic criterion for the bijectivity of Hilbert operators related to Friedrichs systems.
To appear.

On the coupling of hyperbolic and parabolic systems: analytical and numerical approach.