M308: Differential Equations. Mid-Term, November 6th, 2007.
Notes, books, and calculators are not authorized.

Show all your work in the blank space you are given on the exam sheet. **Always justify your answer**. Answers with **no justification will not be graded**.

Useful and useless formulae:

1. $e^{i\theta} = \cos(\theta) + i\sin(\theta)$
2. $\mathfrak{Re}(e^{i\theta}) = \cos(\theta)$, $\mathfrak{Im}(e^{i\theta}) = \sin(\theta)$,
3. $i^2 = -1$
4. $\sin(2x) = 2\sin(x)\cos(x)$
5. $\cos(2x) = \cos^2(x) - \sin^2(x)$
6. $1 = \cos^2(x) + \sin^2(x)$
7. $\frac{2}{(s^2-4s+5)(s-3)} = \frac{-s+1}{s^2-4s+5} + \frac{1}{s-3}$
8. $s^2 - 4s + 5 = (s-2)^2 + 1$
9. $\frac{1}{s(s^2+1)} = \frac{1}{s} - \frac{s}{s^2+1}$
10. $\mathcal{L}(f)(s) = \int_0^{+\infty} e^{-st} f(t)dt$
11. $\mathcal{L}(t^n) = \frac{n!}{s^{n+1}}$, $\mathcal{L}(1) = \frac{1}{s}$, $\mathcal{L}(t) = \frac{1}{s^2}$
12. $h_a(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } 0 < t \end{cases}$
13. $\mathcal{L}(h_a)(s) = \frac{e^{-as}}{s}$
14. $\mathcal{L}^{-1}(e^{-as}\mathcal{L}(f))(t) = f(t-a)h_a(t)$
15. $\mathcal{L}(e^{at}f)(s) = \mathcal{L}(f)(s-a)$
16. $\mathcal{L}(f')(s) = s\mathcal{L}(f)(s) - f(0)$
17. $\mathcal{L}(f'')(s) = s^2\mathcal{L}(f)(s) - sf(0) - f'(0)$
18. $\mathcal{L}(t^n f) = (-1)^n \frac{d^n}{dt^n}\mathcal{L}(f)(s)$

Question 1: Given that $y_1(x) = \frac{1}{4}\sin(2x)$ solves $y'' + 2y' + 4y = \cos(2x)$ and $y_2(x) = \frac{1}{4}x - \frac{1}{8}$ and solves $y'' + 2y' + 4y = x$, find the entire solution set to the following problem:

$$y'' + 2y' + 4y = x + \cos(2x).$$

From linearity we infer that $y_1(x) + y_2(x)$ is a particular solution to the equation. We now look for a set of fundamental solutions to the homogeneous equation. The characteristic equation is $r^2 + 2r + 4 = 0$. The discriminant is $\Delta = 4 - 16 = -12$. The two roots are $r_1 = -1 + i\sqrt{3}$, $r_2 = -1 - i\sqrt{3}$. The two fundamental solutions to the homogeneous equation are $y_3(x) = e^{-x}\cos(\sqrt{3}x)$, $y_4(x) = e^{-x}\sin(\sqrt{3}x)$. Then all solutions to the ODE can be put into the following form

$$y(x) = c_3 y_3(x) + c_4 y_4(x) + y_1(x) + y_2(x),$$

where c_3 and c_4 are arbitrary real numbers.

Question 2: Find a particular solution to $y''(t) - y(t) = 3e^{-t}$ using the variation of parameters method.

We look first for fundamental solutions to the homogeneous equation. The characteristic equation is $r^2 - 1 = 0$. The two roots and $r_1 = 1$ and $r_2 = -1$. The two fundamental solutions are $y_1(t) = e^t$ and $y_2(t) = e^{-t}$. We set $y_p(t) = a_1(t)y_1(t) + a_2(t)y_2(t)$. Then $y_p' = a_1 y_1' + a_2 y_2'$, provided we assume $a_1' y_1 + a_2' y_2 = 0$. Then $y_p'' = a_1' y_1' + a_2' y_2' + a_1 y_1'' + a_2 y_2''$. By inserting y_p into the ODE we obtain $a_1' y_1' + a_2' y_2' = 3e^{-t}$. We now have to solve the following linear system:

$$a_1' e^t + a_2' e^{-t} = 0 \tag{1}$$

$$a_1' e^t - a_2' e^{-t} = 3e^{-t}. \tag{2}$$

Adding the two equations gives $2a_1' e^t = 3e^{-t}$. As a result $a_1 = -\frac{3}{4}e^{-2t}$. This in turns implies $a_2' e^{-t} = -\frac{3}{2}e^{-2t}e^t$. As a result $a_2 = -\frac{3}{2}t$. Then $y_p(t) = -\frac{3}{4}e^{-t} - \frac{3}{2}te^{-t}$. But since $-\frac{3}{4}e^{-t}$ solves the homogeneous system, another particular solution is $z_p(t) = -\frac{3}{2}te^{-t}$.

Question 3: The motion of a pendulum is described by the following ODE: $m\theta''(t)+mgl^{-1}\sin(\theta) = 0$. Show that the quantity $\frac{1}{2}(\theta'(t))^2 - gl^{-1}\cos(\theta(t))$ does not depend on t.

Multiplying the ODE by θ' we obtain

$$\begin{aligned} 0 &= (m\theta''(t) + m\frac{g}{l}\sin(\theta))\theta'(t) \\ &= m\frac{d}{dt}(\theta')\theta' - m\frac{g}{l}\frac{d}{dt}(\cos(\theta)) \\ &= m\frac{d}{dt}(\frac{1}{2}(\theta')^2) - m\frac{g}{l}\frac{d}{dt}(\cos(\theta)) \\ &= m\frac{d}{dt}(\frac{1}{2}(\theta')^2 - \frac{g}{l}\cos(\theta)). \end{aligned}$$

Integrating the above equation yields the desired result.

Question 4: Write the normal form of the following second-order ODE:

$$(1+t^2)y'' + y' - y = 0, \quad y(0) = 1, \quad y'(0) = -1$$

We set $y_1 = y$ and $y_2 = y_1' = y'$. That gives the normal form of the equation:

$$\begin{aligned} &\frac{d}{dt}y_1 = y_2, && y_1(0) = 1 \\ &\frac{d}{dt}y_2 = (1+t^2)^{-1}(y_1 - y_2), && y_2(0) = -1. \end{aligned}$$

Question 5: Use the definition of the Laplace transform to compute the Laplace transform of $\cos(bt)$ and $\sin(bt)$, $b \in \mathbb{R}$. **Do not use a result from a table**. (Hint: use complex notation).

Observing[1] that $\mathcal{L}(\cos(bt)) = \mathfrak{Re}(\mathcal{L}(e^{ibt}))$ and $\mathcal{L}(\sin(bt)) = \mathfrak{Im}(\mathcal{L}(e^{ibt}))$, and assuming $s > 0$, we have

$$\mathcal{L}(e^{ibt}) = \int_0^\infty e^{ibt-st} dt = \frac{1}{ib-s}[e^{(ib-s)t}]_0^{+\infty} = \frac{1}{s-ib} = \frac{s+ib}{s^2+b^2} = \frac{s}{s^2+b^2} + i\frac{b}{s^2+b^2}.$$

As a result we have $\mathcal{L}(\cos(bt)) = s(s^2+b^2)^{-1}$ and $\mathcal{L}(\sin(bt)) = (s^2+b^2)^{-1}$.

[1] Use formula 1 and 2.

Question 6: Solve the following ODE using the Laplace transform method: $y''-4y'+5y = 4e^{3t}$, $y(0) = 2$, $y'(0) = 7$.

Using the properties of the Laplace transform[2] we infer

$$s^2\mathcal{L}(y) - 2s - 7 - 4(s\mathcal{L}(y) - 2) + 5\mathcal{L}(y) = \frac{4}{s-3}.$$

As a result

$$\mathcal{L}(y) = \frac{1}{s^2-4s+5}\left(\frac{4}{s-3} + 2s - 1\right).$$

Then using the given decomposition[3] we deduce

$$\mathcal{L}(y) = \frac{-2s+2}{s^2-4s+5} + \frac{2}{s-3} + \frac{2s-1}{s^2-4s+5} = \frac{1}{s^2-4s+5} + \frac{2}{s-3} = \frac{1}{(s-2)^2+1} + \frac{2}{s-3}.$$

Then taking the inverse Laplace transform[4] we obtain

$$y(t) = e^{2t}\sin(t) + 2e^{3t}.$$

[2] Use formula 15, 16, and 17.
[3] Formula 7.
[4] Use formula 15.

Question 7: Let $h_a(t)$ be the function equal to 0 if $t < a$ and equal to 1 if $t > a$. Solve the following ODE using the Laplace transform method: $y'' + y = h_3(t)$, $y(0) = 0$, $y'(0) = 1$.

Taking the Laplace transform[5] of the equation yields

$$\mathcal{L}(y)(s^2+1) = \mathcal{L}(h_3) + 1 = \frac{e^{-3s}}{s} + 1.$$

Using the given decomposition (formula 9), we obtain

$$\mathcal{L}(y) = \frac{e^{-3s}}{s(s^2+1)} + \frac{1}{s^2+1} = e^{-3s}\left(\frac{1}{s} - \frac{s}{s^2+1}\right) + \frac{1}{s^2+1}.$$

Taking the inverse Laplace transform (formula 14) gives

$$y(t) = h_3(t)(1-\cos(t-3)) + \sin(t).$$

[5] Use formula 13 and 17.

Question 8: Solve the integro-differential equation for $y(t)$: $y(t) + \int_0^t (t-\tau)y(\tau)d\tau = 1$.

The equation can be recast into

$$y(t) + t * y(t) = 1.$$

Taking the Laplace transform of the equation gives

$$\frac{1}{s} = \mathcal{L}(y) + \mathcal{L}(t)\mathcal{L}(y) = \mathcal{L}(y)(1 + \frac{1}{s^2}).$$

This gives

$$\mathcal{L}(y) = \frac{s}{s^2+1}.$$

Taking the inverse Laplace transform gives $y(t) = \cos(t)$.