M308: Differential Equations. Mid-Term, March 8, 2008.
Notes, books, and calculators are not authorized.

Show all your work in the blank space you are given on the exam sheet. Always justify your answer. Answers with **no justification will not be graded**.

Question 1: Does the Existence and Uniqueness Theorem for first-order ODEs implies that the following problem has a unique solution: $y'(x) = 3x - (y(x)-1)^{\frac{1}{3}}$; $\quad y(0) = 1$. (A yes or no answer only will not be graded. Explain.)

The problem can be recast into the form $y' = f(x,y)$, where $f(x,y) = 3x - (y-1)^{\frac{1}{3}}$. $\partial_y f$ is not continuous at $(0,1)$. The theorem does not permit to conclude. The odds are that there is non-uniqueness.

Question 2: Assuming that the velocity of a particle is given by $x'(t) = (t-x)(x^2+t^2)$. Can a particle at $x = 1$ when $t = 2$ reach the location $x = 0$ at any latter time. (A yes or no answer only will not be graded. Explain.)

No. In the stripe $0 \leq x \leq 1$, $2 \geq t$, we have $x'(t) \geq 2 \times 4 = 8$, i.e. the particles that are at $x = 1$ for $t \geq 2$ move away to the right. It is not possible for a particle at $x = 2$ to come back to $x = 1$.

Question 3: Use Euler's method to approximate the solution to $y'(t) = -t/y; \quad y(0) = 0.1$ at $t = 0.2$ using $\Delta t = 0.1$.

Two step suffices. Set $y_0 = 0.1$, $t_0 = 0$ and $y_{n+1} = y_n + \Delta t(-t_n/y_n)$ with $t_{n+1} = t_n + \Delta t$.

$$y_1 = y_0 + \Delta t(-t_0/y_0) = 0.1 - 0 = 0.1; \quad t_1 = 0.1$$
$$y_2 = y_1 + \Delta t(-t_1/y_1) = 0.1 - 0.1\frac{0.1}{0.1} = 0.$$

Hence $y(0.2) \approx 0$.

Question 4: Solve $x^2dx + 2ydy = 0; \quad y(0) = 2$.

This is a separable equation $2yy' = -x^2$. We integrate, $y^2(x) = -\frac{1}{3}x^3 + c$. The constant must be such that $4 = 0 + c$. As a result $y(x) = \pm(2 - \frac{1}{3}x^3)^{\frac{1}{2}}$, but since $y(0)$ is positive, the solution branch that interests us is $y(x) = (2 - \frac{1}{3}x^3)^{\frac{1}{2}}$.

Question 5: Obtain the solutions to the following ODE: $y'(x) = y(x)x^{-1} + 2x + 1$.

The integration factor is $\exp(\int -x^{-1}dx) = \exp(\log(|x^{-1}| + c) = c'x^{-1}$. Multiplying the equation by this factor we obtain $x^{-1}y'(x) - yx^{-2} = 2 + x^{-1}$. This also gives $(yx^{-1})' = 2 + x^{-1}$. We integrate: $y(x) = 2x^2 + x\log|x| + cx$, where c is an arbitrary constant.

Question 6: Is the following differential form exact? $(2xy^3 + 1)dx + (3x^2y^2 - y^{-1})dy = 0$. If yes, give an implicit representation of the solutions.

Set $M(x, y) = (2xy^3 + 1)$ and $N(x, y) = (3x^2y^2 - y^{-1})$. Check for exactness. $\partial_y M(x, y) = 6xy^2$, $\partial_x N(x, y) = 6xy^2$. The form is exact since $\partial_y M(x, y) = \partial_x N(x, y)$. Let us then solve for $F(x, y)$ where $\partial_x F(x, y) = M(x, y)$. This means

$$F(x, y) = x^2y^3 + x + g(y),$$

where $g(y)$ is given by

$$3x^2y^2 + g(y)' = \partial_y F(x, y) = N(x, y) = 3x^2y^2 - y^{-1}$$

In other words,

$$g(y)' = -y^{-1},$$

meaning $g(y) = -\log(|y|) + c$. The solutions are given as implicit solutions to $F(x, y) = 0$ where $F(x, y) = x^2y^3 + x - \log(|y|) + c$, where c is an arbitrary constant.

Question 7: Determine the recursive formulas for the second-order Taylor method for the following ODE: $y'(t) = ty(t) - y^2; \quad y(0) = 1$.

For a general ODE $y'(t) = f(t, y)$, the formula is $y_{n+1} = y_n + \Delta t f(t_n, y_n) + \frac{1}{2}\Delta t^2(\partial_t f(t_n, y_n) + f(t_n, y_n)\partial_y f(t_n, y_n))$. We have $\partial_t f(t_n, y_n) = y_n$ and $\partial_y f(t_n, y_n) = t_n - 2y_n$. As result the algorithm is as follows:

$$y_0 = 1; \quad t_0 = 0$$

$$y_{n+1} = y_n + \Delta t(t_n y_n - y_n^2) + \frac{1}{2}\Delta t^2(y_n + (t_n - 2y_n)(t_n y_n - y_n^2)); \quad t_{n+1} = t_n + \Delta t.$$

Question 8: Solve $y''(t) - 5y'(t) + 6y(t) = 0; \quad y(0) = -1; \quad y'(0) = -4$.

The characteristic equation is $r^2 - 5r + 6 = 0$. The two roots are real $r_1 = 3$, $r_2 = 2$. The solution is of the form $y(t) = c_1 e^{2t} + c_2 e^{3t}$, where $c_1 + c_2 = -1$ and $2c_1 + 3c_2 = -4$. As a result, $c_1 = 1$, $c_2 = -2$ and $y(t) = e^{2t} - 2e^{3t}$.