

**M412: Theory of Partial Differential Equations**  
**Mid-Term TEST, November 14, 2008**  
**Notes, books, and calculators are not authorized.**

Show all your work in the blank space you are given on the exam sheet. Always justify your answer. Answers with **no justification will not be graded.**

Here are some formulae that you may want to use:

$$\mathcal{F}(f)(\omega) \stackrel{\text{def}}{=} \frac{1}{2\pi} \int_{-\infty}^{+\infty} f(x)e^{i\omega x} dx, \quad \mathcal{F}^{-1}(f)(x) = \int_{-\infty}^{+\infty} f(\omega)e^{-i\omega x} d\omega, \quad (1)$$

$$\mathcal{F}(f * g) = 2\pi\mathcal{F}(f)\mathcal{F}(g), \quad (2)$$

$$\mathcal{F}(e^{-\alpha|x|}) = \frac{1}{\pi} \frac{\alpha}{\omega^2 + \alpha^2}, \quad \mathcal{F}\left(\frac{2\alpha}{x^2 + \alpha^2}\right)(\omega) = e^{-\alpha|\omega|}, \quad (3)$$

**Question 1**

Solve the wave equation on the semi-infinite domain  $(0, +\infty)$ ,

$$\partial_{tt}w - 4\partial_{xx}w = 0, \quad x \in (0, +\infty), \quad t > 0$$

$$w(x, 0) = (1 + x^2)^{-1}, \quad x \in (0, +\infty); \quad \partial_t w(x, 0) = 0, \quad x \in (0, +\infty); \quad \text{and} \quad \partial_x w(0, t) = 0, \quad t > 0.$$

(Hint: Consider a particular extension of  $w$  over  $\mathbb{R}$ )

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We define  $f(x) = (1 + x^2)^{-1}$  and its even extension  $f_e(x)$  on  $-\infty, +\infty$ . Let  $w_e$  be the solution to the wave equation over the entire real line with  $f_e$  as initial data:

$$\partial_{tt}w_e - 4\partial_{xx}w_e = 0, \quad x \in \mathbb{R}, \quad t > 0$$

$$w_e(x, 0) = f_e(x), \quad x > 0,$$

$$\partial_t w_e(x, 0) = 0, \quad x \in \mathbb{R}.$$

The solution to this problem is given by the D'Alembert formula

$$w_e(x, t) = \frac{1}{2}(f_e(x - 2t) + f_e(x + 2t)), \quad \text{for all } x \in \mathbb{R} \text{ and } t \geq 0.$$

Let  $x$  be positive. Then  $w(x, t) = w_e(x, t)$  for all  $x \in (0, +\infty)$ , since by construction  $\partial_x w_e(0, t) = 0$  for all times.

Case 1: If  $x - 2t > 0$ ,  $f_e(x - 2t) = f(x - 2t)$ ; as a result

$$w(x, t) = \frac{1}{2}(f(x - 2t) + f(x + 2t)), \quad \text{if } x - 2t > 0.$$

Case 2: If  $x - 2t < 0$ ,  $f_e(x - 2t) = f(-x + 2t)$ ; as a result

$$w(x, t) = \frac{1}{2}(f(-x + 2t) + f(x + 2t)), \quad \text{if } x - 2t < 0.$$

Note that actually  $f_e(x) = (1 + x^2)^{-1}$ ; as a result, the solution can also be re-written as follows:

$$w(x, t) = \frac{1}{2} \left( \frac{1}{1 + (x - 2t)^2} + \frac{1}{1 + (x + 2t)^2} \right).$$


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**Question 2**

Solve the PDE

$$\begin{aligned}
 u_{tt} - u_{xx} &= 0, & 0 \leq x \leq 1, \quad 0 < t, \\
 \partial_x u(0, t) &= 0, \quad \partial_x u(1, t) = 0 & 0 < t, \\
 u(x, 0) &= \cos(\pi x), \quad u_t(x, 0) = 0, & 0 < x < +1.
 \end{aligned}$$

(Hint: Consider the periodic extension over  $\mathbb{R}$  of a particular extension of  $u$  over  $[-1 + 1]$ ).

The even extension of  $u$  over  $[-1 + 1]$ , say  $u_e$ , satisfies the PDE and the initial conditions, and always satisfies  $\partial_x u_e(0, t) = 0$ ,  $\partial_x u_e(1, t) = 0$ . Since  $\partial_x u_e(1, t)$ , we deduce  $\partial_x u_e(-1, t) = 0$ . This means that the periodic extension of  $u_e$ , says  $u_p$ , is smooth and also satisfies the PDE plus the initial conditions. By construction  $\partial_x u_p(0, t) = 0$  and  $\partial_x u_p(1, t) = 0$ . As a result, we can obtain  $u$  by computing the solution of the wave equation on  $\mathbb{R}$  using the periodic extension over  $\mathbb{R}$  of the even extension of the initial data over  $[-1 + 1]$ , i.e.,  $u = u_p|_{[0,1]}$

We have to define the even extension of  $\cos(\pi x)$  on  $(-1, +1)$ . Clearly  $\cos(\pi x)$  is the even extension. Now we define the periodic extension of  $\cos(\pi x)$  over the entire real line. Clearly  $\cos(\pi x)$  is the extension in question. The D'Alembert formula, which is valid on the entire real line, gives

$$\begin{aligned}
 u(x, t) &= \frac{1}{2}(\cos(\pi(x - t)) + \cos(\pi(x + t))) \\
 &= \frac{1}{2}((\cos(\pi t) \cos(\pi x) + \sin(\pi t) \sin(\pi x)) + \frac{1}{2}((\cos(\pi t) \cos(\pi x) - \sin(\pi t) \sin(\pi x))) \\
 &= \cos(\pi t) \cos(\pi x).
 \end{aligned}$$

Hence  $u(x, t) = \cos(\pi t) \cos(\pi x)$  for all  $x \in (0, 1)$ ,  $t > 0$ .

**Question 3**

Let  $\Omega = \{(t, x) \in \mathbb{R}^2 : t > 0, x \geq t\}$ . Let  $\Gamma$  be defined by the following parameterization  $\Gamma = \{x = x_\Gamma(s), t = t_\Gamma(s), s \in \mathbb{R}\}$ , with  $x_\Gamma(s) = -s$  and  $t_\Gamma(s) = -s$  if  $s \leq 0$ ,  $x_\Gamma(s) = s$  and  $t_\Gamma(s) = 0$  if  $s \geq 0$ . Solve the following PDE (give the implicit and explicit representations):

$$u_t + 3u_x + 2u = 0, \quad \text{in } \Omega, \quad u(x, t) = u_\Gamma(x, t) := \begin{cases} 1 & \text{if } t = 0 \\ 2 & \text{if } x = t \end{cases} \quad \text{for all } (x, t) \text{ in } \Gamma.$$

We define the characteristics by

$$\frac{dx(t, s)}{dt} = 3, \quad x(t_\Gamma(s), s) = x_\Gamma(s).$$

This gives  $x(t, s) = x_\Gamma(s) + 3(t - t_\Gamma(s))$ . Upon setting  $\phi(t, s) = u(x(t, s), t)$ , we observe that  $\partial_t \phi(t, s) + 2\phi(t, s) = 0$ , which means

$$\phi(t, s) = ce^{-2t}.$$

The initial condition implies  $\phi(t_\Gamma(s), s) = u_\Gamma(x_\Gamma(s), t_\Gamma(s))$ ; as a result  $c = u_\Gamma(x_\Gamma(s), t_\Gamma(s))e^{2t_\Gamma(s)}$ .

$$\phi(t, s) = u_\Gamma(x_\Gamma(s), t_\Gamma(s))e^{2(t_\Gamma(s)-t)}.$$

The implicit representation of the solution is

$$u(x(t, s), t) = u_\Gamma(x_\Gamma(s), t_\Gamma(s))e^{2(t_\Gamma(s)-t)}, \quad x(t, s) = x_\Gamma(s) + 3(t - t_\Gamma(s)).$$

Now we give the explicit representation.

Case 1: If  $s \leq 0$ ,  $x_\Gamma(s) = -s$ ,  $t_\Gamma(s) = -s$ , and  $u_\Gamma(x_\Gamma(s), t_\Gamma(s)) = 2$ . This means  $x(t, s) = -s + 3(t + s)$  and we obtain  $s = \frac{1}{2}(x - 3t)$ , which means

$$u(x, t) = 2e^{-2(\frac{1}{2}(x-3t)-t)} = 2e^{t-x}, \quad \text{if } x - 3t < 0.$$

Case 2: If  $s \geq 0$ ,  $x_\Gamma(s) = s$ ,  $t_\Gamma(s) = 0$ , and  $u_\Gamma(x_\Gamma(s), t_\Gamma(s)) = 1$ . This means  $x(t, s) = s + 3t$  and we obtain  $s = x - 3t$ , which means

$$u(x, t) = e^{-2t}, \quad \text{if } x - 3t > 0.$$

**Question 4**

Solve the integral equation:  $f(x) + \frac{3}{2} \int_{-\infty}^{+\infty} e^{-|x-y|} f(y) dy = e^{-|x|}$ , for all  $x \in (-\infty, +\infty)$ .

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The equation can be re-written

$$f(x) + \frac{3}{2} e^{-|x|} * f = e^{-|x|}.$$

We take the Fourier transform of the equation and apply the Convolution Theorem (see (2))

$$\mathcal{F}(f) + \frac{3}{2} 2\pi \mathcal{F}(e^{-|x|}) \mathcal{F}(f) = \mathcal{F}(e^{-|x|}).$$

Using (3), we obtain

$$\mathcal{F}(f) + 3\pi \frac{1}{\pi} \frac{1}{1+\omega^2} \mathcal{F}(f) = \frac{1}{\pi} \frac{1}{1+\omega^2},$$

which gives

$$\mathcal{F}(f) \frac{\omega^2 + 4}{1 + \omega^2} = \frac{1}{\pi} \frac{1}{1 + \omega^2}.$$

We then deduce

$$\mathcal{F}(f) = \frac{1}{\pi} \frac{1}{4 + \omega^2} = \frac{1}{2} \mathcal{F}(e^{-2|x|}).$$

Taking the inverse Fourier transform, we finally obtain  $f(x) = \frac{1}{2} e^{-2|x|}$ .

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**Question 5**

Consider the following conservation equation

$$\partial_t \rho + \partial_x(q(\rho)) = 0, \quad x \in (-\infty, +\infty), \quad t > 0, \quad \rho(x, 0) = \rho_0(x) := \begin{cases} 2 & \text{if } x < 0, \\ 1 & \text{if } x > 0, \end{cases}$$

where  $q(\rho) = \rho(2 - \rho)$  (and  $\rho(x, t)$  is the conserved quantity). Solve this problem using the method of characteristics. Do we have a shock or an expansion wave here?

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The characteristics are defined by

$$\frac{dX(t, x_0)}{dt} = q'(\rho) = 2(1 - \rho(X(t, x_0), t)), \quad X(0, x_0) = x_0.$$

Set  $\phi(t) = \rho(X(t, x_0), t)$  and insert in the equation. We obtain that  $\partial_t \phi(t, x_0) = 0$ ; meaning that  $\phi(t, x_0) = \phi(0, x_0)$ , i.e.,  $\rho$  is constant along the characteristics:  $\rho(X(t, x_0), t) = \rho(x_0, 0) = \rho_0(x_0)$ . As a result we can integrate the equation defining the characteristics and we obtain  $X(t, x_0) = 2(1 - \rho_0(x_0))t + x_0$ . The implicit representation of the solution is

$$X(t, x_0) = 2(1 - \rho_0(x_0))t + x_0; \quad \rho(X(t, x_0), t) = \rho_0(x_0)$$

We then have two cases depending whether  $x_0$  is positive or negative.

Case 1:  $x_0 < 0$ , then  $\rho_0(x_0) = 2$  and  $X(t, x_0) = 2(1 - 2)t + x_0 = -2t + x_0$ . This means  $x_0 = X(t, x_0) + 2t$  and

$$\rho(x, t) = 2 \quad \text{if} \quad x < -2t.$$

Case 2:  $x_0 > 0$ , then  $\rho_0(x_0) = 1$  and  $X(t, x_0) = 2(1 - 1)t + x_0 = x_0$ . This means  $x_0 = X(t, x_0)$  and

$$\rho(x, t) = 1 \quad \text{if} \quad 0 < x.$$

We see that there is a gap in the region  $\{-2t < x < 0\}$ . This implies that there is an expansion wave. We have to consider a third case  $x_0 = 0$  and  $\rho_0 \in (1, 2)$ .

Case 3:  $x_0 = 0$ , then  $X(t, x_0) = 2(1 - \rho_0)t$ , i.e.,  $\rho_0 = 1 - \frac{X(t, x_0)}{2t}$ . This means that

$$\rho(x, t) = 1 - \frac{x}{2t}, \quad \text{if} \quad -2t < x < 0.$$


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