

Quiz 1 (Notes, books, and calculators are not authorized)

Show all your work in the blank space you are given on the exam sheet. Always justify your answer. Answers with **no justification will not be graded**.

Question 1: Let $\phi(x, y) = \cos(x + \sin(x - y))$. Compute $\partial_x \phi(x, y)$ and $\partial_y \phi(x, y)$. (Do not try to simplify the results).

We apply the chain rule repeatedly

$$\partial_x \phi(x, y) = -\sin(x + \sin(x - y))(1 + \cos(x - y)).$$

$$\partial_y \phi(x, y) = -\sin(x + \sin(x - y))(-\cos(x - y)).$$

Question 2: Consider the heat equation $\partial_t T - k\partial_{xx}T = f(x)$, $x \in [a, b]$, $t > 0$, with $f(x) = kx$, where $k > 0$. Compute the steady state solution (i.e., $\partial_t T = 0$) assuming the boundary conditions: $-\partial_n T(a) = 0$, $T(b) = 0$ (∂_n is the normal derivative)

At steady state, T does not depend on t and we have $\partial_{xx}T(x) = -x$, which implies $\partial_x T(x) = \alpha - \frac{1}{2}x^2$, and $T(x) = \beta + \alpha x - \frac{1}{6}x^3$, where $\alpha, \beta \in \mathbb{R}$. The two constants α and β are determined by the boundary conditions. $0 = -\partial_n T(a) = \partial_x T(a) = \alpha - \frac{1}{2}a^2$ and $0 = T(b) = \beta + \alpha b - \frac{1}{6}b^3$. We conclude that $\alpha = \frac{1}{2}a^2$ and $\beta = -\alpha b + \frac{1}{6}b^3 = -\frac{1}{2}a^2b + \frac{1}{6}b^3$. In conclusion

$$T(x) = -\frac{1}{2}a^2b + \frac{1}{6}b^3 + \frac{1}{2}a^2x - \frac{1}{6}x^3 = -\frac{1}{2}a^2(b - x) + \frac{1}{6}(b^3 - x^3).$$

Question 3: Consider the equation $\partial_t c(x, t) - \partial_{xx} c(x, t) = x$, where $x \in [0, L]$, $t > 0$, with $c(x, 0) = f(x)$, $-\partial_n c(0, t) = 6$, $-\partial_n c(L, t) = 5$, (∂_n is the normal derivative). Compute $E(t) := \int_0^L c(\xi, t) d\xi$.

We integrate the equation with respect to x over $[0, L]$

$$\int_0^L \partial_t c(\xi, t) d\xi - \int_0^L \partial_{\xi\xi} c(\xi, t) d\xi = \int_0^L \xi d\xi.$$

Using that $\int_0^L \partial_t c(\xi, t) d\xi = \partial_t \int_0^L c(\xi, t) d\xi$ together with the fundamental theorem of calculus, we infer that

$$d_t E(t) - \partial_x c(L, t) + \partial_x c(0, t) = \frac{1}{2} L^2.$$

The boundary conditions $\partial_x c(0, t) = -\partial_n c(0, t) = 6$, $-\partial_x c(L, t) = -\partial_n c(L, t) = 5$ give

$$d_t E(t) + 5 + 6 = \frac{1}{2} L^2.$$

We now apply the fundamental theorem of calculus with respect to t

$$E(t) - E(0) = \int_0^t \partial_\tau E(\tau) d\tau = \left(\frac{1}{2} L^2 - 11\right)t.$$

In conclusion

$$E(t) = \int_0^L f(\xi) d\xi + \left(\frac{1}{2} L^2 - 11\right)t.$$

Question 4: Let $\phi = x^4 - y^4$ (a) Compute $\Delta\phi(x, y)$. (b) Consider the square $\Omega = [0, 1] \times [0, 1]$ and let Γ be the boundary of Ω . Compute $\int_\Gamma \partial_n \phi d\Gamma$.

(a) The definition $\Delta\phi = \partial_{xx}\phi + \partial_{yy}\phi$ implies that

$$\Delta\phi = \partial_{xx}\phi + \partial_{yy}\phi = 12x^2 - 12y^2 = 12(x^2 - y^2).$$

(b) The definition $\Delta\phi = \operatorname{div}(\nabla\phi)$ and the fundamental theorem of calculus (also known as the divergence theorem) imply that

$$\int_\Gamma \partial_n \phi d\Gamma = \int_\Gamma n \cdot \nabla \phi d\Gamma = \int_\Omega \operatorname{div}(\nabla\phi) d\Omega = \int_\Omega \Delta\phi d\Omega = \int_\Omega 12(x^2 - y^2) dx dy = 0.$$