Quizz 2 (Notes, books, and calculators are not authorized)

Show all your work in the blank space you are given on the exam sheet. Always justify your answer. Answers with **no justification will not be graded**.

Question 1: Let ϕ be a non-zero solution to the eigenvalue problem $\partial_x\big((1+x^2)\partial_x\phi(x)\big)+(1-x^2)\phi(x) = \lambda\phi(x)$, $x \in (0,\pi)$, $\phi(\pi) = 0$, $-\partial_x\phi(0) = 0$. Assuming $\lambda \in \mathbb{R}$, determine the sign of $\lambda - 1$ using the energy method.

Multiply the equation by ϕ, integrate over $(0,\pi)$, and apply the fundamental theorem of calculus (i.e. integrate by parts):

$$\begin{aligned}\int_0^\pi (\lambda - 1 + x^2)(\phi(x))^2 \mathsf{d}x &= \int_0^\pi \phi(x)\partial_x\big((1+x^2)\partial_x\phi(x)\big)\mathsf{d}x \\ &= \int_0^\pi \big(\partial_x(\phi(x)(1+x^2)\partial_x\phi(x)) - (1+x^2)(\partial_x\phi(x))^2\big)\,\mathsf{d}x \\ &= (1+\pi^2)\phi(\pi)\partial_x\phi(\pi) - \phi(0)\partial_x\phi(0) - \int_0^\pi (1+x^2)(\partial_x\phi(x))^2\mathsf{d}x \\ &= -\int_0^\pi (1+x^2)(\partial_x\phi(x))^2\mathsf{d}x.\end{aligned}$$

In conclusion

$$(\lambda-1)\int_0^\pi (\phi(x))^2\mathsf{d}x \le \int_0^\pi (\lambda - 1 + x^2)(\phi(x))^2\mathsf{d}x = -\int_0^\pi (1+x^2)(\partial_x\phi(x))^2\mathsf{d}x.$$

Assuming that ϕ is nonzero, we obtain that $\lambda - 1 \le -(\int_0^\pi (1+x^2)(\partial_x\phi(x))^2\mathsf{d}x)/\int_0^\pi(\phi(x))^2\mathsf{d}x \le 0$, i.e., λ is non-positive. If $\lambda = 0$ then $\partial_x\phi = 0$, which implies that ϕ is constant. The other condition $\phi(\pi) = 0$ implies that $\phi = 0$ which contradicts our assumption that ϕ is non-zero. In conclusion $\lambda - 1$ is negative, i.e., $\lambda < 1$.

Question 2: Let $k, f : [-1,+1] \longrightarrow \mathbb{R}$ be such that $k(x) = 3$, $f(x) = -6$ if $x \in [-1,0]$ and $k(x) = 1$, $f(x) = 2$ if $x \in (0,1]$. Consider the boundary value problem $-\partial_x(k(x)\partial_x T(x)) = f(x)$ with $T(-1) = 1$ and $\partial_x T(1) = 1$.
(a) What should be the interface conditions at $x = 0$ for this problem to make sense?

The function T and the flux $k(x)\partial_x T(x)$ must be continuous at $x = 0$. Let T^- denote the solution on $[-1,0]$ and T^+ the solution on $[0,+1]$. One should have $T^-(0) = T^+(0)$ and $k^-(0)\partial_x T^-(0) = k^+(0)\partial_x T^+(0)$, where $k^-(0) = 3$ and $k^+(0) = 1$.

(b) Solve the problem, i.e., find $T(x)$, $x \in [-1,+1]$. Give all the details.

On $[-1,0]$ we have $k^-(x) = 3$ and $f^-(x) = -6$ which implies $-3\partial_{xx}T^-(x) = -6$. This in turn implies $T^-(x) = x^2 + ax + b$. The Dirichlet condition at $x = -1$ implies that $T^-(-1) = 1 = 1 - a + b$. This gives $a = b$ and $T^-(x) = x^2 + bx + b$.

We proceed similarly on $[0,+1]$ and we obtain $-\partial_{xx}T^+(x) = 2$, which implies that $T^+(x) = -x^2 + cx + d$. The Neumann condition at $x = 1$ implies $\partial_x T^+(1) = 1 = -2 + c$. This gives $c = 3$ and $T^+(x) = -x^2 + 3x + d$.

The interface conditions $T^-(0) = T^+(0)$ and $k^-(0)\partial_x T^-(0) = k^+(0)\partial_x T^+(0)$ give $b = d$ and $3b = 3$, respectively. In conclusion $b = 1$, $d = 1$ and

$$T(x) = \begin{cases} x^2 + x + 1 & \text{if } x \in [-1,0], \\ -x^2 + 3x + 1 & \text{if } x \in [0,1]. \end{cases}$$

Question 3: Consider the heat equation $\partial_t u(x,t) - 2\partial_{xx}u(x,t) = 0$, $\partial_x u(0,t) = 0$, $\partial_x u(1,t) = 0$, $u(x,0) = u_0(x)$, $t > 0$, $x \in (0,1)$. The general solution is $u(x,t) = \sum_{n=1}^{\infty} B_n \cos(n\pi x)\mathrm{e}^{-2n^2\pi^2 t}$. Compute the solution corresponding to the initial data $u_0(x) = 3\cos(2\pi x) - 5\cos(3\pi x)$.

The solution contains two terms only, corresponding to $n = 2$ and $n = 3$,

$$u(x,t) = 3\cos(2\pi x)\mathsf{e}^{-16\pi^2 t} - 5\cos(3\pi x)\mathsf{e}^{-18\pi^2 t}.$$

Question 4: Assume that the following equation has a smooth solution: $-\partial_x((1+\sin(x))\partial_x T(x)) - 2\partial_x T(x) + T(x) = \cos(x)$, $T(a) = 1$, $T(b) = \pi$, $x \in [a,b]$, $t > 0$, where $k > 0$. Prove that this solution is unique by using the energy method.

Assume that there are two solutions T_1 and T_2. Let $\phi = T_2 - T_1$. Then

$$-\partial_x((1+\sin(x))\partial_x\phi(x)) - 2\partial_x\phi(x) + \phi(x) = 0, \quad \phi(a) = 0, \quad \phi(b) = 0$$

Multiply the PDE by ϕ, integrate over (a,b), and integrate by parts (i.e. apply the fundamental theorem of calculus):

$$\begin{aligned}
0 &= \int_a^b \big(-\partial_x((1+\sin(x))\partial_x\phi(x))\phi(x) - 2(\partial_x\phi(x))\phi(x) + (\phi(x))^2\big)\mathsf{d}x \\
&= \int_a^b \big(-\partial_x(\phi(x)(1+\sin(x))\partial_x\phi(x)) + (1+\sin(x))(\partial_x\phi(x))^2 - \partial_x\Big(\phi(x)^2\Big) + (\phi(x))^2\big)\mathsf{d}x \\
&= \int_a^b \big((1+\sin(x))(\partial_x\phi(x))^2 + (\phi(x))^2\big)\mathsf{d}x \geq \int_a^b \phi^2(x)\mathsf{d}x,
\end{aligned}$$

This implies $\int_a^b(\phi(x))^2\mathsf{d}x = 0$, i.e., $\phi = 0$, meaning that $T_2 = T_1$.