Quiz 3 (Notes, books, and calculators are not authorized)

Show all your work in the blank space you are given on the exam sheet. Always justify your answer. Answers with **no justification will not be graded**.

**Question 1:** Does any of the following expressions solve the Laplace equation inside the rectangle $0 \le x \le L$, $0 \le y \le H$, with the following boundary conditions $u(0,y) = 0$, $u(L,y) = 3\sinh(\frac{5\pi L}{H})\sin(\frac{5\pi y}{H})$, $u(x,0) = 0$, $u(x,H) = 0$? (justify clearly your answer):

$$u_1(x,y) = 3\cos(\frac{5\pi y}{H})\cosh(\frac{5\pi(x-L)}{H}), \qquad u_2(x,y) = 3\sin(\frac{5\pi y}{H})\cosh(\frac{5\pi(x-L)}{H}),$$
$$u_3(x,y) = 3\cos(\frac{5\pi y}{H})\sinh(\frac{5\pi x}{H}), \qquad u_4(x,y) = 3\sin(\frac{5\pi y}{H})\sinh(\frac{5\pi x}{H}).$$

---

From class, we know that all the above expressions solve the Laplace equation, hence we just need to verify that the boundary conditions are met. We observe that $u_1$ and $u_3$ do not satisfy the Dirichlet boundary conditions $u(x,0) = 0$, $u(x,H) = 0$; therefore $u_1$ and $u_3$ must be discarded.

Both $u_2$ and $u_4$ satisfy that Dirichlet conditions: $u_2(x,0) = 0$, $u_2(x,H) = 0$, and $u_4(x,0) = 0$, $u_4(x,H) = 0$. Now we need to check the two other boundaries.

Note that $u_2$ is such that $u_2(0,y) = 3\sin(\frac{5\pi y}{H})\cosh(\frac{5\pi(-L)}{H}) \neq 0$, which shows that $u_2$ is not the solution to our problem either.

Finally $u_4(0,y) = 3\sin(\frac{5\pi y}{H})\sinh(0) = 0$ and $u_4(L,y) = 3\sin(\frac{5\pi y}{H})\sinh(\frac{5\pi L}{H})$; a result $u_4$ is the solution.

---

**Question 2:** The solution of the equation, $\frac{1}{r}\partial_r(r\partial_r u) + \frac{1}{r^2}\partial_{\theta\theta}u = 0$, inside the domain $D = \{\theta \in [0,\pi],\ r \in [0,2]\}$, subject to the boundary conditions $u(r,0) = 0$, $u(r,\pi) = 0$, $u(2,\theta) = g(\theta)$ is $u(r,\theta) = \sum_{n=1}^{\infty} b_n r^n \sin(n\theta)$. What is the solution corresponding to $g(\theta) = 5\sin(2\theta) + 2\sin(5\theta)$? (Give all the details.)

---

One must have

$$g(\theta) = 5\sin(2\theta) + 2\sin(5\theta) = \sum_{n=1}^{\infty} b_n 2^n \sin(n\theta)$$

Hence, one must have

$$5 = b_2 2^2, \quad \text{and} \quad 2 = b_5 2^5.$$

This means $b_2 = \frac{5}{2^2}$ and $b_5 = \frac{2}{2^5}$ and the other coefficients are zero. The only non-zero terms in the expansion are $b_2 r^2 \sin(2\theta) + b_5 r^5 \sin(5\theta)$. In conclusion

$$u(r,\theta) = 5\frac{r^2}{2^2}\sin(2\theta) + 2\frac{r^5}{2^5}\sin(5\theta).$$

---

**Question 3:** Let $p, q : [-1,+1] \longrightarrow \mathbb{R}$ be smooth functions. Assume that $p(x) \geq 0$ and $q(x) \geq q_0$ for all $x \in [-1,+1]$, where $q_0 \in \mathbb{R}$. Consider the eigenvalue problem $-\partial_x(p(x)\partial_x\phi(x)) + q(x)\phi(x) = \lambda\phi(x)$, supplemented with the boundary conditions $\phi(-1) = 0$ and $\phi(1) = 0$.

(a) Prove that it is necessary that $\lambda \geq q_0$ for a non-zero (smooth) solution, $\phi$, to exist. (Hint: $q_0 \int_{-1}^{+1} \phi^2(x)\mathrm{d}x \leq \int_{-1}^{+1} q(x)\phi^2(x)\mathrm{d}x$.)

---

As usual we use the energy method. Let $(\phi, \lambda)$ be an eigenpair, then

$$\int_{-1}^{+1} (-\partial_x(p(x)\partial_x\phi(x))\phi(x) + q(x)\phi^2(x))\mathsf{d}x = \lambda \int_{-1}^{+1} \phi^2(x)\mathsf{d}x.$$

After integration by parts and using the boundary conditions, we obtain

$$\int_{-1}^{+1} (p(x)\partial_x\phi(x)\partial_x\phi(x) + q(x)\phi^2(x))\mathsf{d}x = \lambda \int_{-1}^{+1} \phi^2(x)\mathsf{d}x.$$

which, using the hint, can also be re-written

$$\int_{-1}^{+1} (p(x)\partial_x\phi(x)\partial_x\phi(x) + q_0\phi^2(x))\mathsf{d}x \leq \lambda \int_{-1}^{+1} \phi^2(x)\mathsf{d}x.$$

Then

$$\int_{-1}^{+1} p(x)(\partial_x\phi(x))^2\mathsf{d}x \leq (\lambda - q_0) \int_{-1}^{+1} \phi^2(x)\mathsf{d}x.$$

Assume that $\phi$ is non-zero, then

$$\lambda - q_0 \geq \frac{\int_{-1}^{+1} p(x)(\partial_x\phi(x))^2\mathsf{d}x}{\int_{-1}^{+1} \phi^2(x)\mathsf{d}x} \geq 0,$$

which proves that it is necessary that $\lambda \geq q_0$ for a non-zero (smooth) solution to exist.

---

(b) Assume that $p(x) \geq p_0 > 0$ for all $x \in [-1,+1]$ where $p_0 \in \mathbb{R}_+$. Show that $\phi = 0$ if $\lambda = q_0$. (Hint: $p_0 \int_{-1}^{+1} \psi^2(x)\mathrm{d}x \leq \int_{-1}^{+1} p(x)\psi^2(x)\mathrm{d}x$.)

---

Assume that $\lambda = q_0$ is an eigenvalue. Then the above computation shows that

$$p_0 \int_{-1}^{+1} (\partial_x\phi(x))^2\mathsf{d}x \leq \int_{-1}^{+1} p(x)(\partial_x\phi(x))^2\mathsf{d}x = 0,$$

which means that $\int_{-1}^{+1}(\partial_x\phi(x))^2\mathsf{d}x = 0$ since $p_0 > 0$. As a result $\partial_x\phi = 0$, i.e., $\phi(x) = c$ where $c$ is a constant. The boundary conditions $\phi(-1) = 0 = \phi(1)$ imply that $c = 0$. In conclusion $\phi = 0$ if $\lambda = q_0$, thereby proving that $(\phi, q_0)$ is not an eigenpair.