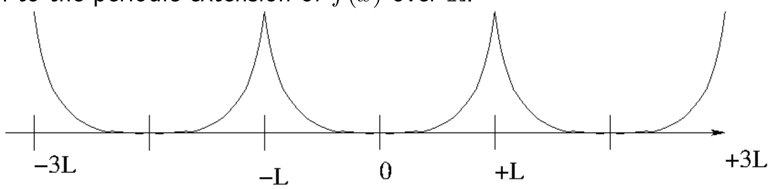
Quiz 4 (Notes, books, and calculators are not authorized)

Show all your work in the blank space you are given on the exam sheet. Always justify your answer. Answers with **no justification will not be graded**.

Question 1: Consider $f : [-L, L] \longrightarrow \mathbb{R}$, $f(x) = x^4$. (a) Sketch the graph of the Fourier series of f and the grah of f.

$FS(f)$ is equal to the periodic extension of $f(x)$ over $\mathbb{R}$.



(b) For which values of $x \in \mathbb{R}$ is $FS(f)$ equal to x^4? (Explain)

The periodic extension of $f(x) = x^4$ over $\mathbb{R}$ is piecewise smooth and globally continuous since $f(L) = f(-L)$. This means that the Fourier series is equal to x^4 over the entire interval $[-L, +L]$.

(c) Is it possible to obtain $FS(x^3)$ by differentiating $\frac{1}{4}FS(x^4)$ term by term? For which values is this legitimate? (Explain)

Yes it is possible since the periodic extension of $f(x) = x^4$ over $\mathbb{R}$ is continuous and piecewise smooth. This operation is legitimate everywhere the function $FS(x^3)$ is smooth, i.e., for all the points in $\mathbb{R}\backslash\{2k-1,\ k \in \mathbb{Z}\}$ (i.e., one needs to exclude the points $\ldots, -7, -5, -3, -1, +1, +3, +5+7, \ldots$

Question 2: Let L be a positive real number. Let $\mathbb{P}_1 = \mathrm{span}\{1, \cos(\pi t/L), \sin(\pi t/L)\}$ and consider the norm $\|f\|_{L^2} := \left(\int_{-L}^{L} f(t)^2 \mathrm{d}t\right)^{\frac{1}{2}}$. (a) Compute the best approximation of $h(t) = 2\cos(\pi t/L) + 7\sin(3\pi t/L)$ in $\mathbb{P}_1$.

The function $h(t) - 2\cos(\pi t/L) = 7\sin(3\pi t/L)$ is orthogonal to all the members of $\mathbb{P}_1$ since the functions $\cos(m\pi t/L)$ and $\sin(m\pi t/L)$ are orthogonal to both $\cos(n\pi t/L)$ and $\sin(n\pi t/L)$ for all $m \neq m$; as a result, the best approximation of h in $\mathbb{P}_1$ is $2\cos(\pi t/L)$. (Recall that the best approximation of h in $\mathbb{P}_1$ is such that $\int_{-L}^{L}(h(t) - FS_1(h))p(t)\mathrm{d}t = 0$ for all $p \in \mathbb{P}_1$.) In conclusion

$$FS_1(h) = 2\cos(\pi t/L).$$

(b) Compute the best approximation of $2 - 3t$ in $\mathbb{P}_1$ with respect to the above norm. (Hint: $\int_{-L}^{L} t\sin(\pi t/L)\mathrm{d}t = 2L^2/\pi$.)

We know from class that the truncated Fourier series

$$FS_1(t) = a_0 + a_1\cos(\pi t/L) + b_1\sin(\pi t/L)$$

is the best approximation. Now we compute a_0, a_1, a_2

$$\begin{aligned}
a_0 &= \frac{1}{2L}\int_{-L}^{L}(2-3t)\mathrm{d}t = 2, \\
a_1 &= \frac{1}{L}\int_{-L}^{L}(2-3t)\cos(\pi t/L)\mathrm{d}t = 0 \\
b_1 &= \frac{1}{L}\int_{-L}^{L}(2-3t)\sin(\pi t/L)\mathrm{d}t = \frac{1}{L}\int_{-L}^{L} -3t\sin(\pi t/L)\mathrm{d}t = -6\frac{L}{\pi} = -\frac{6L}{\pi}.
\end{aligned}$$

As a result

$$FS_1(t) = 2 - \frac{6L}{\pi}\sin(\pi t/L)$$
