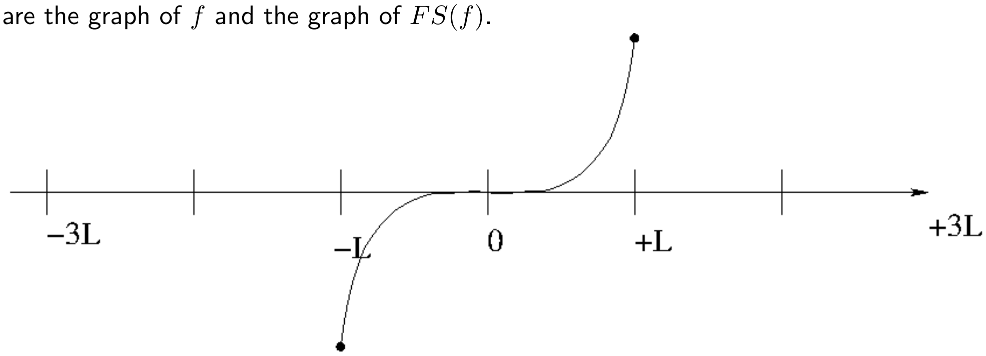
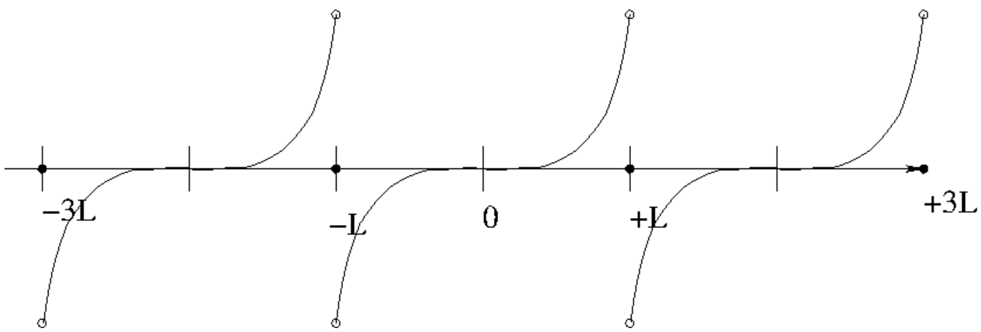
Quiz 4 (Notes, books, and calculators are not authorized)

Show all your work in the blank space you are given on the exam sheet. Always justify your answer. Answers with **no justification will not be graded**.

Question 1: Consider $f : [-L, L] \longrightarrow \mathbb{R}$, $f(x) = |x|x$. (a) Sketch the graph of the Fourier series of f and the grah of f. (Recall that $FS(f)$ is the Fourier series of f.)

$FS(f)$ is equal to the periodic extension of $f(x)$ over $\mathbb{R}$ except at the point $2k+1$, $k \in \mathbb{Z}$, since f is continuous and piecewise of class C^1. Here are the graph of f and the graph of $FS(f)$.





(b) For which values of $x \in \mathbb{R}$ is $FS(f)$ equal to $x|x|$? (Explain)

The periodic extension of $f(x) = |x|$ over $\mathbb{R}$ is piecewise smooth but discontinuous at the points $2k+1$, $k \in \mathbb{Z}$ since $f(L) = f(-L)$. This means that the Fourier series is equal to $|x|x$ over $(-L, L)$ only.

(c) Is it possible to obtain $CS(g(x))$ by differentiating $SS(h(x))$ term by term, where $g : [0, L] \longrightarrow \mathbb{R}$ and $g(x) = x$ and $h : [0, L] \longrightarrow \mathbb{R}$ and $h(x) = \frac{1}{2}x^2$? (Recall that $CS(g)$ is the cosine series of g and $SS(h)$ is the sine series of h.) For which values is this legitimate?

No, this is not possible since the odd extension of $h(x) = \frac{1}{2}x^2$ over $[-L, +L]$ is $h_o(x) = \frac{1}{2}|x|x$, and the periodic extension of $h_o(x)$ is discontinuous at the points in $\{2k-1,\ k \in \mathbb{Z}\}$. Another way to answer this question is to invoke the theorem (seen in class) that says that provided h is of class C^2 over $[0, L]$, $\mathrm{d}SS(h)/\mathrm{d}x = CS(\mathrm{d}h/\mathrm{d}x)$ over $(0, L)$ if and only if $h(0) = h(L)$, which is not the case here.

Question 2: Let L be a positive real number. Let $\mathbb{P}_1 = \text{span}\{1, \cos(\pi t/L), \sin(\pi t/L)\}$ and consider the norm $\|f\|_{L^2} := \left(\int_{-L}^{L} f(t)^2 \mathrm{d}t\right)^{\frac{1}{2}}$. (a) Compute the best approximation of $h(t) = 5\cos(\pi t/L) + 3\sin(2\pi t/L)$ in $\mathbb{P}_1$.

The function $h(t) - 5\cos(\pi t/L) = 3\sin(2\pi t/L)$ is orthogonal to all the members of $\mathbb{P}_1$ since the functions $\cos(m\pi t/L)$ and $\sin(m\pi t/L)$ are orthogonal to both $\cos(n\pi t/L)$ and $\sin(n\pi t/L)$ for all $m \neq m$; as a result, the best approximation of h in $\mathbb{P}_1$ is $5\cos(\pi t/L)$. (Recall that the best approximation of h in $\mathbb{P}_1$ is such that $\int_{-L}^{L}(h(t) - FS_1(h))p(t)\mathsf{d}t = 0$ for all $p \in \mathbb{P}_1$.) In conclusion

$$FS_1(h) = 5\cos(\pi t/L).$$

(b) Compute the best approximation of $-1+5t$ in $\mathbb{P}_1$ with respect to the above norm. (Hint: $\int_{-L}^{L} t\sin(\pi t/L)\mathrm{d}t = 2L^2/\pi$.)

We know from class that the truncated Fourier series

$$FS_1(t) = a_0 + a_1\cos(\pi t/L) + b_1\sin(\pi t/L)$$

is the best approximation. Now we compute a_0, a_1, a_2

$$\begin{aligned}
a_0 &= \frac{1}{2L}\int_{-L}^{L}(-1+5t)\mathsf{d}t = -1, \\
a_1 &= \frac{1}{L}\int_{-L}^{L}(-1+5t)\cos(\pi t/L)\mathsf{d}t = 0 \\
b_1 &= \frac{1}{L}\int_{-L}^{L}(-1+5t)\sin(\pi t/L)\mathsf{d}t = \frac{1}{L}\int_{-L}^{L}5t\sin(\pi t/L)\mathsf{d}t = 10\cos(\pi)\frac{L}{\pi} = \frac{10L}{\pi}.
\end{aligned}$$

As a result

$$FS_1(t) = -1 + \frac{10L}{\pi}\sin(\pi t/L)$$
