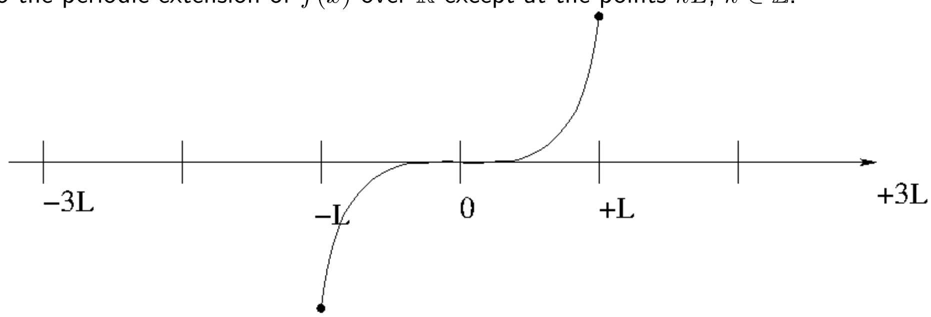
Quiz 4 (Notes, books, and calculators are not authorized)

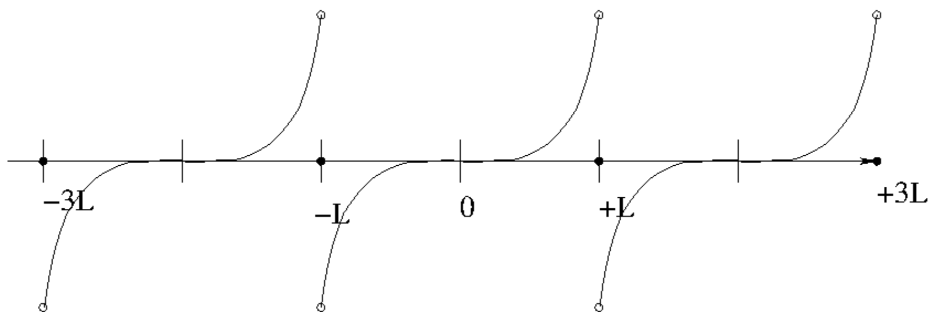
Show all your work in the blank space you are given on the exam sheet. Always justify your answer. Answers with **no justification will not be graded**.

Question 1:

Question 2: Consider $f : [-L, L] \longrightarrow \mathbb{R}$, $f(x) = |x|x$. (a) Sketch the graph of the Fourier series of f and the graph of f.

$FS(f)$ is equal to the periodic extension of $f(x)$ over $\mathbb{R}$ except at the points kL, $k \in \mathbb{Z}$.





(b) For which values of $x \in \mathbb{R}$ is $FS(f)$ equal to $x|x|$? (Explain)

The periodic extension of $f(x) = x|x|$ over $\mathbb{R}$ is smooth over each interval $[(2k-1)L, (2k+1)L]$, $k \in \mathbb{Z}$, but discontinuous at all the points $(2k+1)L$, $k \in \mathbb{Z}$. This means that the Fourier series is equal to $x|x|$ over $(-L, +L)$, (recall that $f(x) = |x|x$ is defined only over $[-L, L]$) and equal to the periodic extension of $x|x|$ over each interval $((2k-1)L, (2k+1)L)$, $k \in \mathbb{Z}$. Since $f(-L) + f(+L) = 0$, the Fourier series is equal to 0 at all the points $(2k+1)L]$, $k \in \mathbb{Z}$.

(c) Is it possible to obtain $FS(2|x|)$ by differentiating $FS(|x|x)$ term by term? (Explain)

No, it is not legitimate since the Fourier series is discontinuous. The result would be wrong.

Question 3: Let L be a positive real number. Let $\mathbb{P}_1 = \text{span}\{1, \cos(\pi t/L), \sin(\pi t/L)\}$ and consider the norm $\|f\|_{L^2} := \left(\int_{-L}^{L} f(t)^2 \mathrm{d}t\right)^{\frac{1}{2}}$. (a) Compute the best approximation of $h(t) = 3 + \pi\cos(\pi t/L) + 3\sin(7\pi t/L)$ in $\mathbb{P}_1$.

The function $h(t) - 3 - \pi\cos(\pi t/L) = 3\sin(7\pi t/L)$ is orthogonal to all the members of $\mathbb{P}_1$ since the functions $\cos(m\pi t/L)$ and $\sin(m\pi t/L)$ are orthogonal to both $\cos(n\pi t/L)$ and $\sin(n\pi t/L)$ for all $m \neq m$; as a result, the best approximation of h in $\mathbb{P}_1$ is $3 + \pi\cos(\pi t/L)$. (Recall that the best approximation of h in $\mathbb{P}_1$ is such that $\int_{-L}^{L}(h(t) - FS_1(h))p(t)\mathrm{d}t = 0$ for all $p \in \mathbb{P}_1$.) In conclusion

$$FS_1(h) = 3 + \pi\cos(\pi t/L).$$

(b) Compute the best approximation of $2 - 3t^2$ in $\mathbb{P}_1$ with respect to the above norm. (Hint: $\int t^2 \cos(t)\mathrm{d}t = 2t\cos(t) + (t^2 - 2)\sin(t)$.)

We know from class that the truncated Fourier series

$$FS_1(t) = a_0 + a_1\cos(\pi t/L) + b_1\sin(\pi t/L)$$

is the best approximation. Now we compute a_0, a_1, a_2

$$a_0 = \frac{1}{2L}\int_{-L}^{L}(2 - 3t^2)\mathrm{d}t = 2 - \frac{2L^3}{2L} = 2 - L^2,$$

$$a_1 = \frac{1}{L}\int_{-L}^{L}(2 - 3t^2)\cos(\pi t/L)\mathrm{d}t = -\frac{3}{L}\frac{L^3}{\pi^3}\int_{-\pi}^{\pi} t^2\cos(t)\mathrm{d}t = -\frac{3}{L}\frac{L^3}{\pi^3}(-4\pi) = 12\frac{L^2}{\pi^2}$$

$$b_1 = \frac{1}{L}\int_{-L}^{L}(2 - 3t^2)\sin(\pi t/L)\mathrm{d}t = 0.$$

As a result

$$FS_1(t) = 2 - L^2 + \frac{12L^2}{\pi^2}\cos(\pi t/L)$$