Quiz 5 (Notes, books, and calculators are not authorized)

Show all your work in the blank space you are given on the exam sheet. Always justify your answer. Answers with **no justification will not be graded**. Here are some formulae that you may want to use:

$$\mathcal{F}(f)(\omega) \stackrel{\text{def}}{=} \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(x)e^{i\omega x}dx, \qquad \mathcal{F}^{-1}(f)(x) = \int_{-\infty}^{+\infty} f(\omega)e^{-i\omega x}d\omega, \tag{1}$$

Question 1: (a) Prove that $\partial_\omega \mathcal{F}(f)(\omega) = i\mathcal{F}(xf(x))(\omega)$ for all $f \in L^1(\mathbb{R})$.

Let $f \in L^1(\mathbb{R})$, then

$$\begin{aligned}\partial_\omega \mathcal{F}(f)(\omega) &= \partial_\omega\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{i\omega x}\mathsf{d}x\right) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\partial_\omega e^{i\omega x}\mathsf{d}x\\ &= i\frac{1}{2\pi}\int_{-\infty}^{\infty} xf(x)e^{i\omega x}\mathsf{d}x,\end{aligned}$$

which prove that $\partial_\omega \mathcal{F}(f)(\omega) = i\mathcal{F}(xf(x))(\omega)$.

(b) Let $\alpha \in \mathbb{R}$ with $\alpha > 0$. Prove that $\mathcal{F}(\partial_x e^{-\alpha x^2})(\omega) = 2\alpha i\partial_\omega \mathcal{F}(e^{-\alpha x^2})(\omega)$. (Hint: use (a).)

We use (a) to deduce that

$$\begin{aligned}\mathcal{F}(\partial_x e^{-\alpha x^2})(\omega) &= \mathcal{F}(-2\alpha x e^{-\alpha x^2})(\omega) = -2\alpha\mathcal{F}(xe^{-\alpha x^2})(\omega)\\ &= 2\alpha i\partial_\omega \mathcal{F}(e^{-\alpha x^2})(\omega).\end{aligned}$$

(c) Show that $\partial_\omega \mathcal{F}(e^{-\alpha x^2})(\omega) = -\frac{\omega}{2\alpha}\mathcal{F}(e^{-\alpha x^2})(\omega)$.

From (b) we have

$$\partial_\omega \mathcal{F}(e^{-\alpha x^2})(\omega) = \frac{1}{2i\alpha}\mathcal{F}(\partial_x e^{-\alpha x^2})(\omega)$$

We now use the property $\mathcal{F}(\partial_x f(x))(\omega) = -i\omega\mathcal{F}(f(x))(\omega)$

$$\partial_\omega \mathcal{F}(e^{-\alpha x^2})(\omega) = \frac{-i\omega}{2i\alpha}\mathcal{F}(e^{-\alpha x^2})(\omega) = \frac{-\omega}{2\alpha}\mathcal{F}(e^{-\alpha x^2})(\omega)$$

which is the desired result.

(d) Given that $\int_{-\infty}^{\infty} e^{-x^2}\mathrm{d}x = \sqrt{\pi}$, compute $\mathcal{F}(e^{-\alpha x^2})(\omega)$. (Hint: Observe that (c) is an ODE and solve it.)

The solution to the ODE $\partial_\omega g(\omega) = -\frac{\omega}{2\alpha}g(\omega)$ is $g(\omega) = g(0)e^{-\frac{\omega^2}{4\alpha}}$. We apply this formula to $g(\omega) = \mathcal{F}(e^{-\alpha x^2})(\omega)$,

$$\mathcal{F}(e^{-\alpha x^2})(\omega) = \mathcal{F}(e^{-\alpha x^2})(0)e^{-\frac{\omega^2}{4\alpha}}.$$

We now need to compute $\mathcal{F}(e^{-\alpha x^2})(0)$,

$$\mathcal{F}(e^{-\alpha x^2})(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\alpha x^2}\mathsf{d}x = \frac{1}{2\pi}\frac{1}{\sqrt{\alpha}}\int_{-\infty}^{\infty} e^{-\alpha x^2}\sqrt{\alpha}\mathsf{d}x = \frac{1}{2\pi}\frac{1}{\sqrt{\alpha}}\int_{-\infty}^{\infty} e^{-x^2}\mathsf{d}x = \frac{1}{2\pi}\frac{\sqrt{\pi}}{\sqrt{\alpha}} = \frac{1}{4\pi\alpha}.$$

Finally

$$\mathcal{F}(e^{-\alpha x^2})(\omega) = \frac{1}{\sqrt{4\pi\alpha}}e^{-\frac{\omega^2}{4\alpha}}.$$

Question 2: State the shift lemma (Do not prove).

Let f be an integrable function over $\mathbb{R}$ (in $L^1(\mathbb{R})$), and let $\beta \in \mathbb{R}$. Using the definitions above, the following holds:

$$\mathcal{F}(f(x-\beta))(\omega) = \mathcal{F}(f)(\omega)e^{i\omega\beta}, \qquad \forall \omega \in \mathbb{R}.$$

Question 3: Use the Fourier transform technique to solve the following PDE:

$$\partial_t u(x,t) + c\partial_x u(x,t) + \gamma u(x,t) = 0,$$

for all $x \in (-\infty, +\infty)$, $t > 0$, with $u(x,0) = u_0(x)$ for all $x \in (-\infty, +\infty)$.

By taking the Fourier transform of the PDE, one obtains

$$\partial_t \mathcal{F}(u) - i\omega c\mathcal{F}(y) + \gamma\mathcal{F}(y) = 0.$$

The solution is

$$\mathcal{F}(u)(\omega,t) = c(\omega)e^{i\omega ct - \gamma t}.$$

The initial condition implies that $c(\omega) = \mathcal{F}(u_0)(\omega)$:

$$\mathcal{F}(u)(\omega,t) = \mathcal{F}(u_0)(\omega)e^{i\omega ct}e^{-\gamma t}.$$

The shift lemma in turn implies that

$$\mathcal{F}(u)(\omega,t) = \mathcal{F}(u_0(x-ct))(\omega)e^{-\gamma t} = \mathcal{F}(u_0(x-ct)e^{-\gamma t})(\omega).$$

Applying the inverse Fourier transform gives:

$$u(x,t) = u_0(x-ct)e^{-\gamma t}.$$