Quiz 6 (Notes, books, and calculators are not authorized)

Show all your work in the blank space you are given on the exam sheet. Always justify your answer. Answers with **no justification will not be graded**.

Question 1: Consider the wave equation $\partial_{tt}w - \partial_{xx}w = 0$, $x \in (0,4)$, $t > 0$, with

$$w(x,0) = f(x), \quad x \in (0,4), \quad \partial_t w(x,0) = 0, \quad x \in (0,4), \quad \text{and} \quad w(0,t) = 0, \quad w(4,t) = 0, \quad t > 0.$$

where $f(x) = x - 1$, if $x \in [1,2]$, $f(x) = 3 - x$, if $x \in [2,3]$, and $f(x) = 0$ otherwise. (a) Give a simple expression of the solution in terms of an extension of f.

We know from class that with Dirichlet boundary conditions, the solution to this problem is given by the D'Alembert formula where f must be replaced by the periodic extension (of period 8) of its odd extension, say $f_{\mathsf{o,p}}$, where

$$f_{\mathsf{o,p}}(x+8) = f_{\mathsf{o,p}}(x), \qquad \forall x \in \mathbb{R}$$

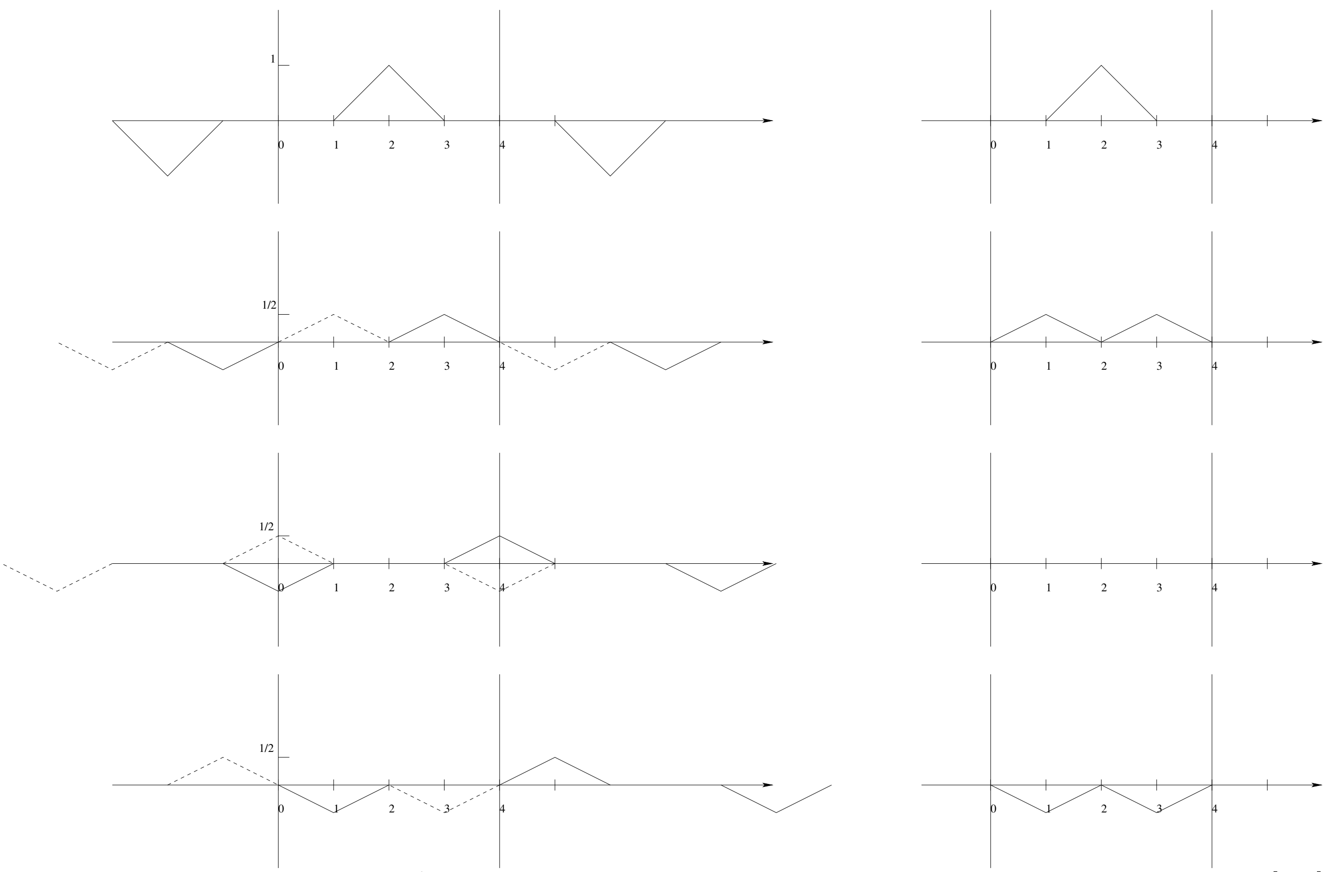
$$f_{\mathsf{o,p}}(x) = \begin{cases} f(x) & \text{if } x \in [0,4] \\ -f(-x) & \text{if } x \in [-4,0) \end{cases}$$

The solution is

$$u(x,t) = \frac{1}{2}(f_{\mathsf{o,p}}(x-t) + f_{\mathsf{o,p}}(x+t)).$$

(b) Give a graphical solution to the problem at $t = 0$, $t = 1$, $t = 2$, and $t = 3$ (draw four different graphs and explain).

I draw on the left of the figure the graph of $f_{\mathsf{o,p}}$. Half the graph moves to the right with speed 1 (solid line in left panel of the figure), the other half moves to the left with speed 1 (dashed line in left panel of the figure). The graph of the solution at times $t = 0, 1, 2, 3$ is shown in the right panel of the figure.



Initial data + periodic extension of the odd extension at $t = 0, 1, 2, 3$.

Solution in domain $[0,4]$ at $t = 0, 1, 2, 3$

Question 2: Consider the equation $\partial_{xx}u(x) = f(x)$, $x \in (0, L)$, with $u(0) = a$ and $\partial_x u(L) = b$.
(a) The Green's function solves $\partial_{xx}G(x, x_0) = \delta_{x_0}$, $G(0, x_0) = 0$, $\partial_x G(L, x_0) = 0$. Compute $G(x, x_0)$.

Let x_0 be a point in $(0, L)$. The Green's function of the problem is such that

$$\partial_{xx}G(x, x_0) = \delta_{x_0}, \quad G(0, x_0) = 0, \quad \partial_x G(L, x_0) = 0.$$

The following holds for all $x \in (0, x_0)$:

$$\partial_{xx}G(x, x_0) = 0.$$

This implies that $G(x, x_0) = ax + b$ in $(0, x_0)$. The boundary condition $G(0, x_0) = 0$ gives $b = 0$. Likewise, the following holds for all $x \in (x_0, L)$:

$$\partial_{xx}G(x, x_0) = 0.$$

This implies that $G(x, x_0) = cx + d$ in (x_0, L). The boundary condition $\partial_x G(L, x_0) = 0$ gives $c = 0$. The continuity of $G(x, x_0)$ at x_0 implies that $ax_0 = d$. The condition

$$\int_{-\epsilon}^{\epsilon} \partial_{xx}G(x, x_0)\mathsf{d}x = 1, \qquad \forall \epsilon > 0,$$

gives the so-called jump condition: $\partial_x G(x_0^+, x_0) - \partial_x G(x_0^-, x_0) = 1$. This means that $0 - a = 1$, i.e., $a = -1$ and $d = -x_0$. In conclusion

$$G(x, x_0) = \begin{cases} -x & \text{if } \leq x \leq x_0, \\ -x_0 & \text{otherwise.} \end{cases}$$

(b) Give the integral representation of u using the Green's function.

Let x_0 be a point in $(0, L)$. The definition of the Dirac measure at x_0 is such that

$$\begin{aligned} u(x_0) &= \langle \delta_{x_0}, u \rangle = \langle \partial_{xx}G(\cdot, x_0), u \rangle \\ &= -\int_0^L \partial_x G(x, x_0)\partial_x u(x)\mathsf{d}x + [\partial_x G(x, x_0)u(x)]_0^L \\ &= \int_0^L G(x, x_0)\partial_{xx}u(x)\mathsf{d}x - [G(x, x_0)\partial_x u(x)]_0^L + [\partial_x G(x, x_0)u(x)]_0^L \\ &= \int_0^L G(x, x_0)f(x)\mathsf{d}x - G(L, x_0)\partial_x u(L) + G(0, x_0)\partial_x u(0) + \partial_x G(L, x_0)u(L) - \partial_x G(0, x_0)u(0). \end{aligned}$$

This finally gives the following representation of the solution:

$$u(x_0) = \int_0^L G(x, x_0)f(x)\mathsf{d}x - G(L, x_0)b - \partial_x G(0, x_0)a$$