

Quiz 6 (Notes, books, and calculators are not authorized)

Show all your work in the blank space you are given on the exam sheet. Always justify your answer. Answers with **no justification will not be graded.**

Question 1: Consider the wave equation $\partial_{tt}w - \partial_{xx}w = 0$, $x \in (0, 4)$, $t > 0$, with

$$w(x, 0) = f(x), \quad x \in (0, 4), \quad \partial_t w(x, 0) = 0, \quad x \in (0, 4), \quad \text{and} \quad w(0, t) = 0, \quad w(4, t) = 0, \quad t > 0.$$

where $f(x) = x - 1$, if $x \in [1, 2]$, $f(x) = 3 - x$, if $x \in [2, 3]$, and $f(x) = 0$ otherwise. (a) Give a simple expression of the solution in terms of an extension of f .

We know from class that with Dirichlet boundary conditions, the solution to this problem is given by the D'Alembert formula where f must be replaced by the periodic extension (of period 8) of its odd extension, say $f_{o,p}$, where

$$f_{o,p}(x+8) = f_{o,p}(x), \quad \forall x \in \mathbb{R}$$

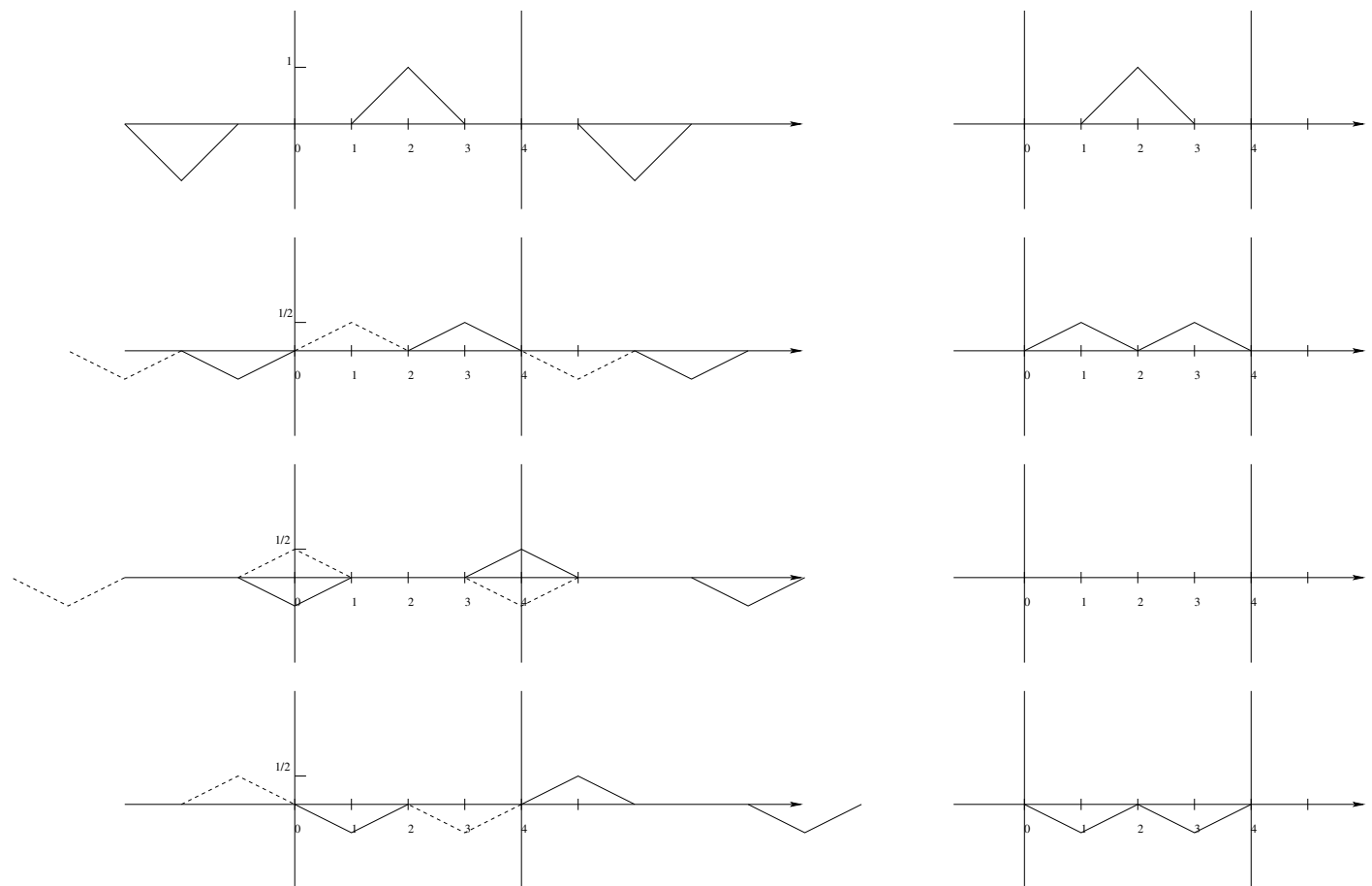
$$f_{o,p}(x) = \begin{cases} f(x) & \text{if } x \in [0, 4] \\ -f(-x) & \text{if } x \in [-4, 0] \end{cases}$$

The solution is

$$u(x, t) = \frac{1}{2}(f_{o,p}(x-t) + f_{o,p}(x+t)).$$

(b) Give a graphical solution to the problem at $t = 0$, $t = 1$, $t = 2$, and $t = 3$ (draw four different graphs and explain).

I draw on the left of the figure the graph of $f_{o,p}$. Half the graph moves to the right with speed 1 (solid line in left panel of the figure), the other half moves to the left with speed 1 (dashed line in left panel of the figure). The graph of the solution at times $t = 0, 1, 2, 3$ is shown in the right panel of the figure.



Initial data + periodic extension of the odd extension at $t = 0, 1, 2, 3$.

Solution in domain $[0, 4]$ at $t = 0, 1, 2, 3$

Question 2: Let $H : \mathbb{R} \rightarrow \mathbb{R}$ be the Heaviside function. (a) Show that the derivative of $H(x) \sin(x)$ in the distribution sense is equal to $H(x) \cos(x)$. (Hint: Compute $-\int_{-\infty}^{\infty} H(x) \sin(x) \partial_x \psi(x) dx$ for any $\psi \in \mathcal{C}_c^1(\mathbb{R})$, integrate by parts ...).

Recall that by definition $\partial_x(H \sin)$ is the distribution that is such that

$$\langle \partial_x(H \sin), \psi \rangle := - \int_{-\infty}^{\infty} H(x) \sin(x) \partial_x \psi(x) dx$$

for all $\psi \in \mathcal{C}_c^1(\mathbb{R})$. We then follow the hint and integrate by parts:

$$\begin{aligned} \langle \partial_x(H \sin), \psi \rangle &= - \int_{-\infty}^{\infty} H(x) \sin(x) \partial_x \psi(x) dx \\ &= - \int_0^{\infty} \sin(x) \partial_x \psi(x) dx = \int_0^{\infty} \cos(x) \psi(x) dx \\ &= \int_{-\infty}^{\infty} H(x) \cos(x) \psi(x) dx. \end{aligned}$$

Hence

$$\langle \partial_x(H \sin), \psi \rangle = \int_{-\infty}^{\infty} H(x) \cos(x) \psi(x) dx, \quad \forall \psi \in \mathcal{C}_c^1(\mathbb{R}).$$

This means that $\partial_x(H(x) \sin(x)) = H(x) \cos(x)$.

(b) Show that the derivative of $H(x) \cos(x)$ in the distribution sense is equal to $-H(x) \sin(x) + \delta_0$ where δ_0 is the Dirac measure at 0. (Hint: Compute $-\int_{-\infty}^{\infty} H(x) \cos(x) \partial_x \psi(x) dx$ for any $\psi \in \mathcal{C}_c^1(\mathbb{R})$, integrate by parts ...).

Recall that by definition $\partial_x(H \cos)$ is the distribution that is such that

$$\langle \partial_x(H \cos), \psi \rangle := - \int_{-\infty}^{\infty} H(x) \cos(x) \partial_x \psi(x) dx$$

for all $\psi \in \mathcal{C}_c^1(\mathbb{R})$. We then follow the hint and integrate by parts:

$$\begin{aligned} \langle \partial_x(H \cos), \psi \rangle &= - \int_{-\infty}^{\infty} H(x) \cos(x) \partial_x \psi(x) dx \\ &= - \int_0^{\infty} \cos(x) \partial_x \psi(x) dx = - \int_0^{\infty} \sin(x) \psi(x) dx + \psi(0) \\ &= \langle \delta_0, \psi \rangle - \int_{-\infty}^{\infty} H(x) \sin(x) \psi(x) dx \\ &= \langle \delta_0 - H \sin, \psi \rangle. \end{aligned}$$

Hence

$$\langle \partial_x(H \cos), \psi \rangle = \langle \delta_0 - H \sin, \psi \rangle, \quad \forall \psi \in \mathcal{C}_c^1(\mathbb{R}).$$

This means that $\partial_x(H(x) \cos(x)) = \delta_0 - H(x) \sin(x)$.