Quiz 6 (Notes, books, and calculators are not authorized)

Show all your work in the blank space you are given on the exam sheet. Always justify your answer. Answers with **no justification will not be graded**. Here is a formula you may want to use,

$$\int_\Omega \psi\Delta\phi \mathrm{d}x = \int_\Omega \phi\Delta\psi \mathrm{d}x + \int_\Gamma \psi\partial_n\phi \mathrm{d}\sigma - \int_\Gamma \phi\partial_n\psi \mathrm{d}\sigma$$

Question 1: Let Ω be a seven-dimensional domain with boundary Γ, and consider the PDE

$$u - \Delta u = f(x), \quad x \in \Omega, \quad \text{with} \quad \partial_n u(x) + 2u(x) = h(x) \quad \text{on the boundary } \Gamma.$$

Let $G(x, x_0)$ be the Green's function of this problem (the exact expression of $G(x, x_0)$ does not matter; just assume that $G(x, x_0)$ is known). (a) what is the PDE solved by $G(x, x_0)$?

Either you remember from class that the operator $u \longmapsto u - \Delta u$ with zero Neumann boundary condition is self-adjoint, or you redo the computation. Multiply the PDE by $G(x, x_0)$, integrate over Ω, and integrate by parts (using the hint):

$$\begin{aligned}
\int_\Omega G(x,x_0)f(x)\mathsf{d}x &= \int_\Omega G(x,x_0)(u(x) - \Delta u(x))\mathsf{d}x \\
&= \int_\Omega G(x,x_0)u(x)\mathsf{d}x - \int_\Omega u(x)\Delta G(x,x_0)\mathsf{d}x - \int_\Gamma G(x,x_0)\partial_n u(x)\mathsf{d}\sigma + \int_\Gamma u(x)\partial_n G(x,x_0)\mathsf{d}\sigma \\
&= \int_\Omega (G(x,x_0) - \Delta G(x,x_0))u(x)\mathsf{d}x + \int_\Gamma G(x,x_0)(2u(x) - h(x))\mathsf{d}\sigma + \int_\Gamma u(x)\partial_n G(x,x_0)\mathsf{d}\sigma \\
&= \int_\Omega (G(x,x_0) - \Delta G(x,x_0))u(x)\mathsf{d}x + \int_\Gamma u(x)(2G(x,x_0) + \partial_n G(x,x_0))\mathsf{d}\sigma - \int_\Gamma G(x,x_0)h(x)\mathsf{d}\sigma
\end{aligned}$$

We then define $G(x, x_0)$ so that

$$G(x,x_0) - \Delta G(x,x_0) = \delta_{x-x_0}, \quad 2G(x,x_0) + \partial_n G(x,x_0)|_\Gamma = 0,$$

where δ_{x-x_0} is the Dirac measure: $\int \delta_{x-x_0}\varphi = \varphi(x_0)$ for all $\varphi \in \mathcal{C}^0(\mathbb{R}^7)$. This means that

$$\int_\Omega G(x,x_0)f(x)\mathsf{d}x = u(x_0) - \int_\Gamma G(x,x_0)h(x)\mathsf{d}\sigma.$$

(b) Give a representation of $u(x)$ in terms of G, f and h.

The above computation shows that

$$u(x_0) = \int_\Omega f(x)G(x,x_0)\mathsf{d}x + \int_\Gamma h(x)G(x,x_0)\mathsf{d}x.$$

Question 2: Consider the equation $\partial_{xx}u(x) = f(x)$, $x \in (0, L)$, with $u(0) = a$ and $\partial_x u(L) = b$.
(a) Compute the Green's function of the problem.

Let x_0 be a point in $(0, L)$. The Green's function of the problem is such that

$$\partial_{xx}G(x, x_0) = \delta_{x_0}, \quad G(0, x_0) = 0, \quad \partial_x G(L, x_0) = 0.$$

The following holds for all $x \in (0, x_0)$:

$$\partial_{xx}G(x, x_0) = 0.$$

This implies that $G(x, x_0) = ax+b$ in $(0, x_0)$. The boundary condition $G(0, x_0) = 0$ gives $b = 0$. Likewise, the following holds for all $x \in (x_0, L)$:

$$\partial_{xx}G(x, x_0) = 0.$$

This implies that $G(x, x_0) = cx + d$ in (x_0, L). The boundary condition $\partial_x G(L, x_0) = 0$ gives $c = 0$. The continuity of $G(x, x_0)$ at x_0 implies that $ax_0 = d$. The condition

$$\int_{-\epsilon}^{\epsilon} \partial_{xx}G(x, x_0)\mathsf{d}x = 1, \qquad \forall \epsilon > 0,$$

gives the so-called jump condition: $\partial_x G(x_0^+, x_0) - \partial_x G(x_0^-, x_0) = 1$. This means that $0 - a = 1$, i.e., $a = -1$ and $d = -x_0$. In conclusion

$$G(x, x_0) = \begin{cases} -x & \text{if } \leq x \leq x_0, \\ -x_0 & \text{otherwise.} \end{cases}$$

(b) Give the integral representation of u using the Green's function.

Let x_0 be a point in $(0, L)$. The definition of the Dirac measure at x_0 is such that

$$\begin{aligned} u(x_0) &= \langle \delta_{x_0}, u \rangle = \langle \partial_{xx}G(\cdot, x_0), u \rangle \\ &= -\int_0^L \partial_x G(x, x_0)\partial_x u(x)\mathsf{d}x + [\partial_x G(x, x_0)u(x)]_0^L \\ &= \int_0^L G(x, x_0)\partial_{xx}u(x)\mathsf{d}x - [G(x, x_0)\partial_x u(x)]_0^L + [\partial_x G(x, x_0)u(x)]_0^L \\ &= \int_0^L G(x, x_0)f(x)\mathsf{d}x - G(L, x_0)\partial_x u(L) + G(0, x_0)\partial_x u(0) + \partial_x G(L, x_0)u(L) - \partial_x G(0, x_0)u(0). \end{aligned}$$

This finally gives the following representation of the solution:

$$u(x_0) = \int_0^L G(x, x_0)f(x)\mathsf{d}x - G(L, x_0)b - \partial_x G(0, x_0)a$$
