M602: Methods and Applications of Partial Differential Equations
Mid-Term TEST, March 8, 2008
Notes, books, and calculators are not authorized.

Show all your work in the blank space you are given on the exam sheet. Always justify your answer. Answers with **no justification will not be graded**.

Question 1

We want to solve the following PDE:

$$\partial_t w + 3\partial_x w = 0, \quad x > -t,\ t > 0$$

$$w(x,t) = w_\Gamma(x,t), \text{ for all } (x,t) \in \Gamma \text{ where}$$

$$\Gamma = \{(x,t) \in \mathbb{R}^2 \text{ s.t. } x = -t,\ x < 0\} \cup \{(x,t) \in \mathbb{R}^2 \text{ s.t. } t = 0,\ x \geq 0\}$$

and w_Γ is a given function.

(a) Draw a picture of the domain Ω where the PDE must be solved, of the boundary Γ, and of the characteristics.

(b) Define a one-to-one parametric representation of the boundary Γ.

For negative s we set $x_\Gamma(s) = s$ and $t_\Gamma(s) = -s$; clearly we have $x_\Gamma(s) = -t_\Gamma(s)$ for all $s < 0$. For positive s we set $x_\Gamma(s) = s$ and $t_\Gamma(s) = 0$. The map $\mathbb{R} \in s \mapsto (x_\Gamma(s), t_\Gamma(s)) \in \Gamma$ is one-t-one.

(c) Give a parametric representation of the characteristics associated with the PDE.

(i) We use α and s to parameterize the characteristics. The characteristics are defined by

$$\frac{dx}{d\alpha}(\alpha, s) = 3, \quad \text{with} \quad x(0,s) = x_\Gamma(s),$$

$$\frac{dt}{d\alpha}(\alpha, s) = 1, \quad \text{with} \quad t(0,s) = t_\Gamma(s).$$

This yields the following parametric representation of the characteristics

$$x(\alpha, s) = 3\alpha + x_\Gamma(s),$$

$$t(\alpha, s) = \alpha + t_\Gamma(s),$$

where $\alpha \geq 0$ and $s \in (-\infty, +\infty)$.

(ii) Another possibility consists of using t to parameterize the characteristics. Then

$$\frac{dx}{dt}(t, s) = 3, \quad \text{with} \quad x(t_\Gamma(s), s) = x_\Gamma(s).$$

This yields

$$x(t, s) = 3(t - t_\Gamma(s)) + x_\Gamma(s).$$

(d) Give a parametric representation of the solution to the PDE.

(i) Now we set $\phi(\alpha, s) = w(x(\alpha, s), t(\alpha, s))$ and we insert this ansatz in the equation. This gives $\frac{d\phi}{d\alpha}(\alpha, s) = 0$, i.e., $\phi(\alpha, s)$ does not depend on α. In other words

$$w(x(\alpha, s), t(\alpha, s)) = \phi(\alpha, s) = \phi(0, s) = w(x(0, s), t(0, s)) = w_\Gamma(x_\Gamma(s), t_\Gamma(s))$$

A parametric representation of the solution is given by

$$\begin{aligned} &x(\alpha, s) = 3\alpha + x_\Gamma(s), \\ &t(\alpha, s) = \alpha + t_\Gamma(s), \\ &w(x(\alpha, s), t(\alpha, s)) = w_\Gamma(x_\Gamma(s), t_\Gamma(s)). \end{aligned}$$

(ii) If we use the second parameterization, we set $\phi(t, s) = w(x(t, s), t)$. Then $\frac{d\phi}{dt}(t, s) = 0$, i.e., $\phi(t, s)$ does not depend on t. In other words

$$w(x(t, s), t) = \phi(t, s) = \phi(t_\Gamma(s), s) = w(x(t_\Gamma, s), t_\Gamma(s)) = w_\Gamma(x_\Gamma(s), t_\Gamma(s))$$

A parametric representation of the solution is given by

$$\begin{aligned} &x(t, s) = 3(t - t_\Gamma(s)) + x_\Gamma(s). \\ &w(x(t, s), t) = w_\Gamma(x_\Gamma(s), t_\Gamma(s)). \end{aligned}$$

(e) Give an explicit representation of the solution.

(i) We have to find the inverse map $(x, t) \mapsto (\alpha, s)$. Clearly $x - 3t = x_\Gamma(s) - 3t_\Gamma(s)$. Then, there are two cases depending on the sign of s.
case 1: If $s < 0$, then $x_\Gamma(s) = s$ and $t_\Gamma(s) = -s$. That means $x - 3t = 4s$, which in turns implies $s = \frac{1}{4}(x - 3t)$. Then

$$w(x, t) = w_\Gamma(\tfrac{1}{4}(x - 3t), -\tfrac{1}{4}(x - 3t)), \quad \text{if } x - 3t < 0.$$

case 2: If $s \geq 0$, then $x_\Gamma(s) = s$ and $t_\Gamma(s) = 0$. That means $x - 3t = s$. Then

$$w(x, t) = w_\Gamma(x - 3t, 0), \quad \text{if } x - 3t \geq 0.$$

(ii) If we use the second parameterization, we have to find the inverse map $(x, t) \mapsto (s, t)$. Clearly $x - 3t = x_\Gamma(s) - 3t_\Gamma(s)$ and we proceed as above. Note that the explicit representation of the solution does not depend on our choice of parameterization.

Question 2

Consider the following conservation equation

$$\partial_t \rho + \partial_x(q(\rho)) = 0, \quad x \in (-\infty, +\infty),\ t > 0, \qquad \rho(x,0) = \rho_0(x) := \begin{cases} \frac{1}{2} & \text{if } x < 0, \\ 1 & \text{if } x > 0, \end{cases}$$

where $q(\rho) = \rho(2-\rho)$ (and $\rho(x,t)$ is the conserved quantity). Solve this problem using the method of characteristics. Do we have a shock or an expansion wave here?

The characteristics are defined by

$$\frac{dX(t)}{dt} = q'(\rho) = 2(1 - \rho(x(t),t)), \quad X(0) = X_0.$$

Set $\phi(t) = \rho(X(t),t)$, then we obtain that ϕ is constant, i.e., ρ is constant along the characteristics: $\rho(X(t),t) = \rho(X_0,0) = \rho_0(X_0)$. As a result we can integrate the equation defining the characteristics and we obtain $X(t) = 2(1-\rho_0(X_0))t + X_0$. We then have two cases depending whether X_0 is positive or negative.

1. $X_0 < 0$, then $\rho_0(X_0) = \frac{1}{2}$ and $X(t) = t + X_0$. This means
$$\rho(x,t) = \frac{1}{2} \quad \text{if} \quad x < t.$$
2. $X_0 > 0$, then $\rho_0(X_0) = 1$ and $X(t) = X_0$. This means
$$\rho(x,t) = 1 \quad \text{if} \quad x > 0.$$

We see that the characteristics cross in the region $\{t > x > 0\}$. This implies that there is a shock. The Rankin-Hugoniot relation gives the speed of this shock:

$$s = \frac{q^+ - q^-}{\rho^+ - \rho^-} = \frac{\frac{3}{4} - 1}{\frac{1}{2} - 1} = \frac{1}{2}.$$

In conclusion

$$\rho = \frac{1}{2}, \quad x < \frac{t}{2},$$
$$\rho = 1, \quad x > \frac{t}{2}.$$

Question 3

Solve the PDE

$$\begin{aligned} &u_{tt} - a^2 u_{xx} = 0, && -\infty < x < +\infty, \quad 0 \le t, \\ &u(x,0) = \cos(x), \quad u_t(x,0) = -a\sin(x), && -\infty < x < +\infty. \end{aligned}$$

Apply D'Alembert's Formula.

$$\begin{aligned} u(x,t) &= \tfrac{1}{2}(\cos(x-at) + \cos(x+at)) - \tfrac{1}{2a}\int_{x-at}^{x+at} a\sin(\xi)d\xi \\ &= \tfrac{1}{2}(\cos(x-at) + \cos(x+at)) + \tfrac{1}{2}(\cos(x+at) - \cos(x-at)) \\ &= \cos(x+at). \end{aligned}$$

Hence $u(x,t) = \cos(x+at)$.

Question 4

Solve the PDE

$$\begin{aligned} & u_{tt} - u_{xx} = 0, && 0 < x < +\infty, \quad 0 < t, \\ & u(0,t) = 0, \quad u(1,t) = 0 && 0 < t, \\ & u(x,0) = \sin(\pi x), \quad u_t(x,0) = 0, && 0 < x < +\infty. \end{aligned}$$

We have to define the odd extension of $\sin(\pi x)$ on $(-1,+1)$. Clearly $\sin(\pi x)$ is the odd extension. Now we define the periodic extension of $\sin(\pi x)$ over the entire real line. Clearly $\sin(\pi x)$ is the extension in question. The D'Alembert formula, which is valid on the entire real line, gives

$$\begin{aligned} u(x,t) &= \tfrac{1}{2}(\sin(\pi(x-t)) + \sin(\pi(x+t)) \\ &= \tfrac{1}{2}((\cos(\pi t)\sin(\pi x) - \sin(\pi t)\cos(\pi x)) + \tfrac{1}{2}((\cos(\pi t)\sin(\pi x) + \sin(\pi t)\cos(\pi x)) \\ &= \cos(\pi t)\sin(\pi x). \end{aligned}$$

Hence $u(x,t) = \cos(\pi t)\sin(\pi x)$ for all $x \in (0,1)$, $t > 0$.

Question 5

Solve the PDE (note that the width and the height of the rectangle are not equal)

$$
\begin{array}{ll}
\partial_{xx}u + \partial_{yy}u = 0, & 0 < x < 1,\ 0 < y < 2, \\
u(x,0) = 8\sin(9\pi x),\ u(x,2) = 0, & 0 < x < 1, \\
u(0,y) = \sin(2\pi y), u(1,y) = 0, & 0 < y < 2.
\end{array}
$$

The method of separation of variables tells us that the solution is a sum of terms like $\sin(n\pi x)\sinh(n\pi(y-2))$ and $\sin(m\pi y/2)\sinh(m\pi(x-1)/2)$. By looking at the boundary conditions we infer that there are two nonzero terms in the expansion: one corresponding to $n = 9$ and one corresponding to $m = 4$. This gives

$$u(x,y) = 8\sin(9\pi x)\frac{\sinh(9\pi(2-y))}{\sinh(18\pi))} + \sin(2\pi y)\frac{\sinh(2\pi(1-x))}{\sinh(2\pi))}$$

Question 6

Let u be a continuous function on $\overline{D}$ where D is some open, connect set in $\mathbb{R}^2$. Explain why, if u is harmonic, it is generally a waste of time to locate a point where u achieves its maximum by solving $\partial_x u = 0$ and $\partial_y u = 0$ simultaneously.

From the Maximum Principle, we know that if u is not constant, the maximum of u is achieved at the boundary of D. The zero gradient condition does not apply for maximums at the boundary.