A

Banach and Hilbert Spaces

The goal of this appendix is to restate some fundamental results on Banach and Hilbert spaces. The emphasis is set on the characterization of bijective Banach operators. The results collected herein provide a theoretical framework for the mathematical analysis of the finite element method. Most of classical results are stated without proof; see [Aub00, Bre91, OdD96, Rud66, Sho96, Yos80, Zei95] for further insight.

A.1 Basic Definitions and Results

For the sake of simplicity, we consider real vector spaces. All the material presented herein can be easily extended to complex vector spaces.

A.1.1 Norm and scalar product

Let V be a real vector space.

Definition A.1 (Norm). *A* norm *on V is a mapping*

$$\|\cdot\|_V : V \ni v \longmapsto \|v\|_V \in \mathbb{R}_+,$$

satisfying the three following properties:

(i) $\|v\|_V = 0 \iff v = 0$.
(ii) $\forall c \in \mathbb{R}$, $\forall v \in V$, $\|cv\|_V = |c|\,\|v\|_V$.
(iii) *Triangle inequality:* $\forall v, w \in V$, $\|v + w\|_V \leq \|v\|_V + \|w\|_V$.

Moreover, a seminorm *on V is a mapping from V to $\mathbb{R}_+$ which satisfies only properties* (ii) *and* (iii).

Definition A.2 (Equivalent norms). *Two norms $\|\cdot\|_{V,1}$ and $\|\cdot\|_{V,2}$ are said to be* equivalent *if there exist two positive constants c_1 and c_2 such that*

$$\forall v \in V, \qquad c_1\|v\|_{V,2} \leq \|v\|_{V,1} \leq c_2\|v\|_{V,2}.$$

Remark A.3. If the normed space V is finite-dimensional, all the norms in V are equivalent. This result is false in infinite-dimensional normed spaces. □

Definition A.4 (Inner product). *An* inner product *or* scalar product *on V is a bilinear mapping*

$$(\cdot,\cdot)_V : V \times V \ni (v,w) \longmapsto (v,w)_V \in \mathbb{R},$$

satisfying the three following properties:

(i) *Symmetry:* $\forall v, w \in V,\ (v,w)_V = (w,v)_V$.
(ii) *Positivity:* $\forall v \in V,\ (v,v)_V \geq 0$.
(iii) $(v,v)_V = 0 \iff v = 0$.

Proposition A.5. *Let $(\cdot,\cdot)_V$ be a scalar product on V. By setting*

$$\forall v \in V, \qquad \|v\|_V = (v,v)_V^{\frac{1}{2}},$$

one defines a norm on V. Moreover, the Cauchy–Schwarz inequality *holds:*

$$\forall v, w \in V, \qquad (v,w)_V \leq \|v\|_V \|w\|_V. \tag{A.1}$$

Remark A.6. The Cauchy–Schwarz[1] inequality can be seen as a consequence of the identity

$$\|v\|_V \|w\|_V - (v,w)_V = \tfrac{\|v\|_V \|w\|_V}{2} \left\| \tfrac{v}{\|v\|_V} - \tfrac{w}{\|w\|_V} \right\|_V^2,$$

valid for all non-zero v and w in V. This identity clearly shows that equality holds in (A.1) if and only if v and w are collinear. □

Proposition A.7 (Arithmetic–geometric inequality). *Let $x_1, \ldots, x_n$ be non-negative numbers. Then,*

$$(x_1 x_2 \ldots x_n)^{\frac{1}{n}} \leq \tfrac{1}{n}(x_1 + \ldots + x_n). \tag{A.2}$$

This inequality is frequently used in conjunction with the Cauchy–Schwarz inequality. In particular, it implies

$$\forall \gamma > 0,\ \forall v, w \in V, \quad (v,w)_V \leq \tfrac{\gamma}{2}\|v\|_V^2 + \tfrac{1}{2\gamma}\|w\|_V^2. \tag{A.3}$$

A.1.2 Operators in Banach spaces

Definition A.8 (Banach spaces). Banach spaces *are complete normed vector spaces. This means that a Banach space is a vector space V equipped with a norm $\|\cdot\|_V$ such that every Cauchy sequence (with respect to the metric $d(x,y) = \|x-y\|_V$) in V has a limit in V.*

[1] Augustin-Louis Cauchy (1789–1857) and Herman Schwarz (1843–1921)

Definition A.9. *Let V and W be two normed vector spaces. $\mathcal{L}(V;W)$ is the vector space of continuous linear mappings from V to W. The mapping A is also called an* operator.

Proposition A.10. *Let V be a normed vector space and let W be a Banach space. Equip $\mathcal{L}(V;W)$ with the norm*

$$\forall A \in \mathcal{L}(V;W), \quad \|A\|_{\mathcal{L}(V;W)} = \sup_{v \in V} \frac{\|Av\|_W}{\|v\|_V}.$$

Then, $\mathcal{L}(V;W)$ is a Banach space.

Proof. See [Rud66, p. 87] or [Yos80, p. 111]. □

Remark A.11. In this book, we shall systematically abuse the notation by writing $\sup_{v \in V} \frac{\|Av\|_W}{\|v\|_V}$ instead of $\sup_{\substack{v \in V \\ v \neq 0}} \frac{\|Av\|_W}{\|v\|_V}$. □

Definition A.12 (Compact operator). *Let V and W be two Banach spaces. $A \in \mathcal{L}(V;W)$ is called a* compact operator *if from every bounded sequence $\{v_n\}_{n \geq 0}$ in V, one can extract a subsequence $\{v_{n_k}\}$ such that the sequence $\{Av_{n_k}\}$ converges in W.*

Remark A.13. If $V \subset W$ and the injection of V into W is compact, then from every bounded sequence $\{v_n\}$ in V, one can extract a subsequence that converges in W. □

A.1.3 Duality

Definition A.14 (Dual space, Continuous linear forms). *Let V be a normed vector space. The* dual space *of V is defined to be $\mathcal{L}(V;\mathbb{R})$ and is denoted by V'. An element $A \in V'$ is called a* continuous linear form. *Its action on an element $v \in V$ is often denoted by means of duality brackets $\langle \cdot, \cdot \rangle_{V',V}$ so that $\langle A, v \rangle_{V',V} = Av$.*

Remark A.15. Owing to Proposition A.10, V' is a Banach space when equipped with the norm

$$\forall A \in V', \quad \|A\|_{V'} = \sup_{v \in V} \frac{\langle A, v \rangle_{V',V}}{\|v\|_V}.$$ □

Theorem A.16 (Hahn–Banach). *Let V be a normed vector space and let G be a subspace of V equipped with the same norm. Let $B \in G' = \mathcal{L}(G;\mathbb{R})$ be a linear continuous mapping with norm*

$$\|B\|_{G'} = \sup_{g \in G} \frac{\langle B, g \rangle_{G',G}}{\|g\|_V}.$$

Then, there exists $A \in V'$ with the following properties:

(i) *A is an extension of B, i.e., $Ag = Bg$ for all $g \in G$.*
(ii) $\|A\|_{V'} = \|B\|_{G'}$.

Proof. See [Rud66, p. 56], [Yos80, p. 102], or [Bre91, p. 4]. The statement is actually a simplified version of the Hahn–Banach Theorem. ☐

Corollary A.17. *Let V be a normed vector space. For all $v \in V$,*

$$\|v\|_V = \sup_{A \in V', \|A\|_{V'}=1} |\langle A, v\rangle_{V',V}| = \max_{A \in V', \|A\|_{V'}=1} |\langle A, v\rangle_{V',V}|.$$

Proof. Assume $v \neq 0$. Clearly, $\sup_{A \in V', \|A\|_{V'}=1} |\langle A, v\rangle_{V',V}| \leq \|v\|_V$. Let $G = \text{span}(v)$ and $B \in G'$ defined as $\langle B, tv\rangle_{G',G} = t\|v\|_V^2$ for $t \in \mathbb{R}$. Owing to the Hahn–Banach Theorem, there exists $Z \in V'$ such that $\|Z\|_{V'} = \|B\|_{G'} = \|v\|_V$ and $\langle Z, v\rangle_{V',V} = \|v\|_V^2$. Conclude by taking $A = \|v\|_V^{-1} Z$. ☐

Corollary A.18. *Let V be a normed space and let $F \subset V$ be subspace. Assume $(\forall f \in V',\ f(F) = 0) \Rightarrow (f = 0)$. Then, $\overline{F} = V$.*

Proof. See [Rud66, Theorem 5.19] or [Bre91, p. 7]. ☐

Definition A.19 (Dual operator). *Let V and W be two normed vector spaces and let $A \in \mathcal{L}(V; W)$. The* dual operator *$A^T : W' \to V'$ is defined by*

$$\forall v \in V,\ \forall w' \in W', \quad \langle A^T w', v\rangle_{V',V} = \langle w', Av\rangle_{W',W}.$$

Definition A.20 (Continuous bilinear forms). *Let Z_1 and Z_2 be two normed vector spaces. $\mathcal{L}(Z_1 \times Z_2; \mathbb{R})$ denotes the vector space of* continuous bilinear forms *on $Z_1 \times Z_2$. It is a Banach space when equipped with the norm*

$$\|a\|_{Z_1,Z_2} = \sup_{z_1 \in Z_1, z_2 \in Z_2} \frac{a(z_1, z_2)}{\|z_1\|_{Z_1} \|z_2\|_{Z_2}}.$$

Proposition A.21. *Let Z_1 and Z_2 be two Banach spaces and let $a \in \mathcal{L}(Z_1 \times Z_2; \mathbb{R})$. Then, the mapping $A : Z_1 \to Z_2'$ defined by*

$$\forall z_1 \in Z_1,\ \forall z_2 \in Z_2, \quad \langle Az_1, z_2\rangle_{Z_2',Z_2} = a(z_1, z_2),$$

is in $\mathcal{L}(Z_1; Z_2')$ and $\|A\|_{\mathcal{L}(Z_1;Z_2')} = \|a\|_{Z_1,Z_2}$.

Definition A.22 (Double dual). *The* double dual *of a Banach space V is the dual of V' and is denoted by V''.*

Remark A.23. Owing to Proposition A.10, V'' is a Banach space. ☐

Proposition A.24. *Let V be a Banach space and let $J_V : V \to V''$ be the linear mapping defined by*

$$\forall u \in V,\ \forall v' \in V', \quad \langle J_V u, v'\rangle_{V'',V'} = \langle v', u\rangle_{V',V}.$$

Then, J_V is an isometry.

Proof. J_V is an isometry since

$$\|J_V u\|_{V''} = \sup_{\substack{v' \in V' \\ \|v'\|_{V'}=1}} \langle J_V u, v' \rangle_{V'',V'} = \sup_{\substack{v' \in V' \\ \|v'\|_{V'}=1}} \langle v', u \rangle_{V',V} = \|u\|_V,$$

where the last equality results from Corollary A.17. □

Remark A.25.

(i) Since the mapping J_V is an isometry, it is injective. As a result, V can be identified with the subspace $J_V(V) \subset V''$.

(ii) It may happen that the mapping J_V is not surjective. In this case, the space V is strictly included in V''. For instance, $L^\infty(\Omega) = L^1(\Omega)'$ but $L^1(\Omega) \subsetneq L^\infty(\Omega)'$ with strict inclusion; see §B.1.2 or [Bre91, pp. 63–66] for the definition of these spaces. □

Definition A.26 (Reflexive Banach spaces). *Let V be a Banach space. V is said to be* reflexive *if J_V is an isomorphism.*

A.1.4 Hilbert spaces

Definition A.27 (Hilbert spaces). *A* Hilbert space *is an inner product space that is complete with respect to the norm defined by the inner product (and is hence a Banach space). A Hilbert space is said to be* separable *if it admits a countable and dense subset.*

Theorem A.28 (Riesz–Fréchet). *Let V be a Hilbert space. For each $v' \in V'$, there exists a unique $u \in V$ such that*

$$\forall w \in V, \quad \langle v', w \rangle_{V',V} = (u, w)_V.$$

Moreover, the mapping $v' \in V' \mapsto u \in V$ is an isometric isomorphism.

Proof. See [Yos80, p. 90] or [Bre91, p. 81]. □

An important consequence of the Riesz–Fréchet Theorem is the following:

Proposition A.29. *Hilbert spaces are reflexive.*

Proof. Let V be a Hilbert space. The Riesz–Fréchet Theorem implies that V can be identified with V'; similarly, V' can be identified with V''. □

Definition A.30 (Orthogonal projection). *Let H be a Hilbert space with scalar product $(\cdot,\cdot)_H$ and associated norm $\|\cdot\|_H$. Let S be a closed subspace of H. The* orthogonal projection *from H to S is defined to be the operator $P_{H,S} \in \mathcal{L}(H;S)$ such that*

$$\forall w \in S, \quad (P_{H,S}(v), w)_H = (v, w)_H. \tag{A.4}$$

Proposition A.31. *$P_{H,S}$ is characterized by the property*

$$\forall v \in H, \quad \|v - P_{H,S}(v)\|_H = \min_{w \in S} \|v - w\|_H. \tag{A.5}$$

Proof. See Exercise **??**. □

A.2 Bijective Banach Operators

This section presents classical results to characterize bijective linear Banach operators [Aub00, Bre91, Yos80]. Some of the material presented herein is adapted from [Aze95, GuQ97]. Henceforth, V and W are real Banach spaces, and operators in $\mathcal{L}(V;W)$ are called *Banach operators.*

A.2.1 Fundamental results

For $A \in \mathcal{L}(V;W)$, we denote by $\mathrm{Ker}(A)$ its kernel and by $\mathrm{Im}(A)$ its range. The operator A being continuous, $\mathrm{Ker}(A)$ is closed in V. Hence, the quotient of V by $\mathrm{Ker}(A)$, $V/\mathrm{Ker}(A)$, can be defined. This space is composed of equivalence classes $\overline{v}$ such that v and w are in the same class $\overline{v}$ if and only if $v-w \in \mathrm{Ker}(A)$.

Theorem A.32. *Equipped with the norm* $\|\overline{v}\| = \inf_{v\in\overline{v}} \|v\|_V$*,* $V/\mathrm{Ker}(A)$ *is a Banach space. Moreover, defining* $\overline{A} : V/\mathrm{Ker}(A) \to \mathrm{Im}(A)$ *by* $\overline{A}\overline{v} = Av$ *for all* v *in* $\overline{v}$*,* $\overline{A}$ *is an isomorphism.*

Proof. See [Yos80, p. 60]. □

For $M \subset V$, $N \subset V'$, we introduce the so-called *annihilator* of M and N,

$$M^\perp = \{v' \in V'; \forall m \in M,\ \langle v', m\rangle_{V',V} = 0\},$$
$$N^\perp = \{v \in V; \forall n' \in N,\ \langle n', v\rangle_{V',V} = 0\}.$$

A characterization of $\mathrm{Ker}(A)$ and $\mathrm{Im}(A)$ is given by the following:

Lemma A.33. *For* A *in* $\mathcal{L}(V;W)$*, the following properties hold:*

(i) $\mathrm{Ker}(A) = (\mathrm{Im}(A^T))^\perp$.
(ii) $\mathrm{Ker}(A^T) = (\mathrm{Im}(A))^\perp$.
(iii) $\overline{\mathrm{Im}(A)} = (\mathrm{Ker}(A^T))^\perp$.
(iv) $\overline{\mathrm{Im}(A^T)} \subset (\mathrm{Ker}(A))^\perp$.

Proof. See [Yos80, pp. 202–209] or [Bre91, p. 28]. □

Lemma A.33(i) shows that the characterization of operators with closed range is important to characterize surjective operators. This is the purpose of the following fundamental theorem:

Theorem A.34 (Banach or Closed Range). *Let* $A \in \mathcal{L}(V;W)$*. The following statements are equivalent:*

(i) $\mathrm{Im}(A)$ *is closed.*
(ii) $\mathrm{Im}(A^T)$ *is closed.*
(iii) $\mathrm{Im}(A) = (\mathrm{Ker}(A^T))^\perp$.
(iv) $\mathrm{Im}(A^T) = (\mathrm{Ker}(A))^\perp$.

Proof. See [Yos80, p. 205] or [Bre91, p. 29]. □

We now put in place the second keystone of the edifice:

Theorem A.35 (Open Mapping). *If $A \in \mathcal{L}(V;W)$ is surjective and U is an open set in V, then $A(U)$ is open in W.*

Proof. See [Rud66, pp. 47–48], [Yos80, p. 75], or [Bre91, p. 18]. □

Theorem A.35, also due to Banach, has far-reaching consequences. In particular, we deduce the following:

Lemma A.36. *Let $A\in\mathcal{L}(V;W)$. The following statements are equivalent:*

(i) $\mathrm{Im}(A)$ *is closed.*
(ii) *There exists $\alpha > 0$ such that*

$$\forall w \in \mathrm{Im}(A),\ \exists v_w \in V, \quad Av_w = w \quad \textit{and} \quad \alpha\|v_w\|_V \le \|w\|_W. \tag{A.6}$$

Proof. The implication (i) $\Rightarrow$ (ii). Since $\mathrm{Im}(A)$ is closed in W, $\mathrm{Im}(A)$ is a Banach space. Applying the Open Mapping Theorem to $A : V \to \mathrm{Im}(A)$ and $U = B_V(0,1)$ (the unit ball in V) yields that $A(B_V(0,1))$ is open in $\mathrm{Im}(A)$. Since $0 \in A(B_V(0,1))$, there is $\gamma > 0$ such that $B_W(0,\gamma) \subset A(B_V(0,1))$. Let $w \in \mathrm{Im}(A)$. Since $\frac{\gamma}{2}\frac{w}{\|w\|_W} \in B_W(0,\alpha)$, there is $z \in B_V(0,1)$ such that $Az = \frac{\gamma}{2}\frac{w}{\|w\|_W}$. In other words, setting $v = \frac{2\|w\|_W}{\gamma}z$, $Av = w$ and $\frac{\gamma}{2}\|v\|_V \le \|w\|_W$.
The implication (ii) $\Rightarrow$ (i). Let $\{w_n\}$ be a sequence in $\mathrm{Im}(A)$ that converges to some $w \in W$. Using (A.6), we infer that there exists a sequence $\{v_n\}$ in V such that $Av_n = w_n$ and $\alpha\|v_n\|_V \le \|w_n\|_W$. Since $\{v_n\}$ is a Cauchy sequence in V and V is a Banach space, v_n converges to a certain $v \in V$. Owing to the continuity of A, Av_n converges to Av. Hence, $w = Av \in \mathrm{Im}(A)$, proving statement (i). □

Remark A.37. A first consequence of Lemma A.36 is that if $A \in \mathcal{L}(V;W)$ is bijective, then its inverse is necessarily continuous. Indeed, the fact that A is bijective implies that A is injective and $\mathrm{Im}(A)$ is closed. Lemma A.36 implies that there is $\alpha > 0$ such that $\|A^{-1}w\|_V \le \frac{1}{\alpha}\|w\|_W$, i.e., A^{-1} is continuous. □

Let us finally give a sufficient condition for the image of an injective operator to be closed.

Lemma A.38 (Petree–Tartar). *Let X, Y, Z be three Banach spaces. Let $A \in \mathcal{L}(X;Y)$ be an injective operator and let $T \in \mathcal{L}(X;Z)$ be a compact operator. If there is $c > 0$ such that $c\|x\|_X \le \|Ax\|_Y + \|Tx\|_Z$, then $\mathrm{Im}(A)$ is closed; equivalently, there is $\alpha > 0$ such that*

$$\forall x \in X, \qquad \alpha\|x\|_X \le \|Ax\|_Y.$$

Proof. By contradiction. Assume that there is a sequence $\{x_n\}$ of X such that $\|x_n\|_X = 1$ and $\|Ax_n\|_Y$ converges to zero when n goes to infinity. Since T is compact and the sequence $\{x_n\}$ is bounded, there is a subsequence $\{x_{n_k}\}$ such that $\{Tx_{n_k}\}$ is a Cauchy sequence in Z. Owing to the inequality

$$\alpha\|x_{n_k} - x_{m_k}\|_X \leq \|Ax_{n_k} - Ax_{m_k}\|_Y + \|Tx_{n_k} - Tx_{m_k}\|_Z,$$

$\{x_{n_k}\}$ is a Cauchy sequence in X. Let x be the limit of the subsequence $\{x_{n_k}\}$ in X. The continuity of A implies $Ax_{n_k} \to Ax$ and $Ax = 0$ since $Ax_{n_k} \to 0$. Since A is injective $x = 0$, which contradicts the hypothesis $\|x_{n_k}\|_X = 1$. □

A.2.2 Characterization of surjective operators

As a consequence of the Closed Range Theorem together with Lemma A.36, which is a rephrasing of the Open Mapping Theorem, we deduce two lemmas characterizing surjective operators. The proofs are left as an exercise.

Lemma A.39. *Let $A \in \mathcal{L}(V;W)$. The following statements are equivalent:*

(i) $A^T : W' \to V'$ *is surjective.*
(ii) $A : V \to W$ *is injective and* $\mathrm{Im}(A)$ *is closed in* W.
(iii) *There exists* $\alpha > 0$ *such that*

$$\forall v \in V, \qquad \|Av\|_W \geq \alpha\|v\|_V. \tag{A.7}$$

(iv) *There exists* $\alpha > 0$ *such that*

$$\inf_{v\in V} \sup_{w'\in W'} \frac{\langle w', Av\rangle_{W',W}}{\|w'\|_{W'}\|v\|_V} \geq \alpha. \tag{A.8}$$

Lemma A.40. *Let $A \in \mathcal{L}(V;W)$. The following statements are equivalent:*

(i) $A : V \to W$ *is surjective.*
(ii) $A^T : W' \to V'$ *is injective et* $\mathrm{Im}(A^T)$ *is closed in* V'.
(iii) *There exists* $\alpha > 0$ *such that*

$$\forall w' \in W', \qquad \|A^T w'\|_{V'} \geq \alpha\|w'\|_{W'}. \tag{A.9}$$

(iv) *There exists* $\alpha > 0$ *such that*

$$\inf_{w'\in W'} \sup_{v\in V} \frac{\langle A^T w', v\rangle_{V',V}}{\|w'\|_{W'}\|v\|_V} \geq \alpha. \tag{A.10}$$

Remark A.41. The statement (i) ⇔ (iv) in Lemma A.40 is sometimes referred to as Lions' Theorem; see, e.g., [Sho96, LiM68]. Establishing the a priori estimate (A.10) is a necessary and sufficient condition to prove that the problem $Au = f$ has at least one solution u in V for all f in W. □

One easily verifies (see Lemma A.42) that (A.6) implies the *inf-sup condition* (A.10). In practice, however, it is often easier to check condition (A.10) than to prove that for all $w \in \text{Im}(A)$, there exists an inverse image v_w satisfying (A.6). At this point, the natural question that arises is to determine whether the constant α in (A.10) is the *same* as that in (A.6). The answer to this question is the purpose of the next lemma which is due to Azerad [Aze95, Aze99]. This lemma will be used in the study of saddle-point problems; see Theorem **??**.

Lemma A.42. *Let V and W be two Banach spaces and let $A \in \mathcal{L}(V;W)$ be a surjective operator. Let $\alpha > 0$. The property*

$$\forall w \in \text{Im}(A),\ \exists v_w \in V, \quad Av_w = w \quad \textit{and} \quad \alpha\|v_w\|_V \le \|w\|_W,$$

implies

$$\inf_{w' \in W'} \sup_{v \in V} \frac{\langle A^T w', v\rangle_{V',V}}{\|w'\|_{W'}\|v\|_V} \ge \alpha.$$

The converse is true if V is reflexive.

Proof. (1) The implication. By definition of the norm in W',

$$\forall w' \in W', \qquad \|w'\|_{W'} = \sup_{\substack{w \in W \\ \|w\|_W \le 1}} \langle w', w\rangle_{W',W}.$$

For all w in W, there is $v_w \in V$ such that $Av_w = w$ and $\alpha\|v_w\|_V \le \|w\|_W$. Let w' in W'. Therefore,

$$\langle w', w\rangle_{W',W} = \langle w', Av_w\rangle_{W',W} = \langle A^T w', v_w\rangle_{V',V} \le \tfrac{1}{\alpha}\|A^T w'\|_{V'}\|w\|_W.$$

Hence,

$$\|w'\|_{W'} = \sup_{\substack{w \in W \\ \|w\|_W \le 1}} \langle w', w\rangle_{W',W} \le \tfrac{1}{\alpha}\|A^T w'\|_{V'}.$$

The desired inequality follows from the definition of the norm in V'.
(2) Let us prove the converse statement by assuming that V is reflexive. The inf-sup inequality being equivalent to $\|A^T w'\|_{V'} \ge \alpha\|w'\|_{W'}$ for all $w' \in W'$, A^T is injective. Let $v' \in \text{Im}(A^T)$ and define $z'(v') \in W'$ such that $A^T(z'(v')) = v'$. Note that $z'(v')$ is unique since A^T is injective. Hence, $z'(\cdot) : \text{Im}(A^T) \subset V' \to W'$ is a mapping. Let $w \in W$ and let us construct an inverse image for w, say v_w, satisfying (A.6). We first define the linear form $\tilde{w} : \text{Im}(A^T) \to \mathbb{R}$ by

$$\forall v' \in \text{Im}(A^T), \quad \tilde{w}(v') = \langle z'(v'), w\rangle_{W',W},$$

that is, $\tilde{w}(A^T z') = \langle z', w\rangle_{W',W}$ for all $z' \in W'$. Hence,

$$\begin{aligned} |\tilde{w}(v')| \le \|z'(v')\|_{W'}\|w\|_W &\le \tfrac{1}{\alpha}\|A^T z'(v')\|_{V'}\|w\|_W \\ &\le \tfrac{1}{\alpha}\|v'\|_{V'}\|w\|_W. \end{aligned}$$

This means that $\tilde{w}$ is continuous on $\mathrm{Im}(A^T)$ equipped with the norm of V'. Owing to the Hahn–Banach Theorem, $\tilde{w}$ can be extended to V' with the same norm. Let $\tilde{\tilde{w}} \in V''$ be the extension in question with $\|\tilde{\tilde{w}}\|_{V''} \leq \frac{1}{\alpha}\|w\|_W$. Since V is assumed to be reflexive, there is $v_w \in V$ such that $J_V(v_w) = \tilde{\tilde{w}}$. As a result,

$$\forall z' \in W', \qquad \langle z', Av_w\rangle_{W',W} = \langle A^T z', v_w\rangle_{V',V} = \langle J_V(v_w), A^T z'\rangle_{V'',V'}$$
$$= \langle \tilde{\tilde{w}}, A^T z'\rangle_{V'',V'} = \langle z', w\rangle_{W',W},$$

showing that $Av_w = w$. Hence, v_w is an inverse image of w and

$$\|v_w\|_V = \|J_V(v_w)\|_{V''} = \|\tilde{\tilde{w}}\|_{V''} \leq \tfrac{1}{\alpha}\|w\|_W. \qquad \square$$

A.2.3 Characterization of bijective Banach operators

Theorem A.43. *Let $A \in \mathcal{L}(V;W)$. A is bijective if and only if $A^T : W' \to V'$ is injective and there exists $\alpha > 0$ such that*

$$\forall v \in V, \qquad \|Av\|_W \geq \alpha\|v\|_V. \tag{A.11}$$

Proof. (1) The implication. Since A is surjective, $\mathrm{Ker}(A^T) = \mathrm{Im}(A)^\perp = \{0\}$, i.e., A^T is injective. Since $\mathrm{Im}(A) = W$ is closed and A is injective, we deduce from Lemma A.39 that there exists $\alpha > 0$ such that $\|Av\|_W \geq \alpha\|v\|_V$.
(2) The converse. The injectivity of A^T implies $\overline{\mathrm{Im}(A)} = (\mathrm{Ker}(A^T))^\perp = W$, i.e., $\mathrm{Im}(A)$ is dense in W. Let us prove that $\mathrm{Im}(A)$ is closed. Let $\{v_n\}$ be a sequence in V such that $\{Av_n\}$ is a Cauchy sequence in W. The inequality $\|Av_n\|_W \geq \alpha\|v_n\|_V$ implies that $\{v_n\}$ is a Cauchy sequence in V. Let v be its limit. The continuity of A implies $Av_n \to Av$; hence, $\mathrm{Im}(A)$ is closed. Therefore, $\mathrm{Im}(A) = W$, i.e., A is surjective. Finally, the injectivity of A is a direct consequence of inequality (A.11). $\square$

Remark A.44. The interpretation of Theorem A.43 is that a Banach operator is bijective if and only if it is injective, its range is closed, and its dual operator is injective. $\square$

Corollary A.45. *Let $A \in \mathcal{L}(V;W)$. The following statements are equivalent:*

(i) *A is bijective.*
(ii) *There exists a constant $\alpha > 0$ such that*

$$\forall v \in V, \quad \|Av\|_W \geq \alpha\|v\|_V, \tag{A.12}$$
$$\forall w' \in W', \quad (A^T w' = 0) \implies (w' = 0). \tag{A.13}$$

(iii) *There exists a constant $\alpha > 0$ such that*

$$\inf_{v\in V} \sup_{w'\in W'} \frac{\langle w', Av\rangle_{W',W}}{\|w'\|_{W'}\|v\|_V} \geq \alpha, \tag{A.14}$$
$$\forall w' \in W', \quad (\langle w', Av\rangle_{W',W} = 0,\ \forall v \in V) \implies (w' = 0). \tag{A.15}$$

Proof. Condition (A.13) is equivalent to stating that A^T is injective. Therefore, Theorem A.43 shows that the bijectivity of A is equivalent to the conditions of statement (ii). Furthermore, statements (ii) and (iii) are clearly equivalent since the inf-sup condition (A.14) is a simple reformulation of (A.12) and since (A.13) and (A.15) are clearly equivalent. □

Now, let us assume that $A \in \mathcal{L}(V;W)$ is associated with a bilinear form $a \in \mathcal{L}(Z_1 \times Z_2;\mathbb{R})$ such that $\langle Az_1, z_2\rangle_{Z_2',Z_2} = a(z_1,z_2)$, i.e., $V = Z_1$ and $W = Z_2'$.

Corollary A.46. *If Z_2 is reflexive, the following statements are equivalent:*

(i) *For all $f \in Z_2'$, there is a unique $u \in Z_1$ such that $a(u,z_2) = \langle f, z_2\rangle_{Z_2',Z_2}$ for all $z_2 \in Z_2$.*
(ii) *There is $\alpha > 0$ such that*

$$\inf_{z_1\in Z_1}\sup_{z_2\in Z_2}\frac{a(z_1,z_2)}{\|z_1\|_{Z_1}\|z_2\|_{Z_2}} \geq \alpha, \tag{A.16}$$

$$\forall z_2 \in Z_2, \quad (\forall z_1 \in Z_1,\, a(z_1,z_2) = 0) \implies (z_2 = 0). \tag{A.17}$$

Proof. Item (i) amounts to stating that A is bijective. Owing to Corollary A.45, the bijectivity of A is equivalent to conditions (A.12) and (A.13). Clearly, (A.12) is equivalent to $\sup_{z_2\in Z_2}\frac{a(z_1,z_2)}{\|z_2\|_{Z_2}} \geq \alpha\|z_1\|_{Z_1}$, which is (A.16). Furthermore, (A.15) is clearly equivalent to (A.17) since Z_2 is reflexive. □

Remark A.47. Corollary A.46 is the BNB Theorem; see §**??**. □

A.2.4 Coercive operators

In this section, we focus on a smaller class of operators, that of coercive operators.

Definition A.48 (Coercive operator). *$A \in \mathcal{L}(V;V')$ is said to be a* coercive operator *if there is a constant $\alpha > 0$ such that*

$$\forall v \in V, \qquad \langle Av, v\rangle_{V',V} \geq \alpha\|v\|_V^2.$$

The following proposition shows that the notion of coercivity is relevant only in *Hilbert spaces*:

Proposition A.49. *Let V be a Banach space. V can be equipped with a Hilbert structure with the same topology if and only if there is a coercive operator in $\mathcal{L}(V;V')$.*

Proof. See Exercise **??**. □

Corollary A.50. *Coercivity is a sufficient condition for an operator $A \in \mathcal{L}(V;V')$ to be bijective.*

Proof. See Lemma **??**. □

Remark A.51. Corollary A.50 is the *Lax–Milgram Lemma*; see §**??**. □

We now introduce the class of self-adjoint operators.

Definition A.52 (Self-adjoint operator). *Let V be a reflexive Banach space, so that V and V'' are identified. $A \in \mathcal{L}(V;V')$ is said to be a* self-adjoint operator *if $A^T = A$.*

Self-adjoint bijective operators are characterized as follows:

Corollary A.53. *Let V be a reflexive Banach space and let $A \in \mathcal{L}(V;V')$ be a self-adjoint operator. Then, A is bijective if and only if there is $\alpha > 0$ such that*

$$\forall v \in V, \qquad \|Av\|_{V'} \geq \alpha \|v\|_V. \tag{A.18}$$

Proof. Owing to Theorem A.43, the bijectivity of A implies that A satisfies inequality (A.18). Conversely, inequality (A.18) means that A is injective. It follows that A^T is injective since $A^T = A$ by hypothesis. The conclusion is then a consequence of Theorem A.43. □

We finally introduce the concept of monotonicity.

Definition A.54 (Monotone operator). *$A \in \mathcal{L}(V;V')$ is said to be* monotone operator *if*

$$\forall v \in V, \qquad \langle Av, v\rangle_{V',V} \geq 0.$$

Corollary A.55. *Let V be a reflexive Banach space and let $A \in \mathcal{L}(V;V')$ be a monotone self-adjoint operator. Then, A is bijective if and only if A is coercive.*

Proof. See Exercise **??**. □

A.2.5 Application to saddle-point problems

Let X and M be two real Banach spaces. Consider two continuous linear operators $A : X \to X'$ and $B : X \to M$, and let $B^T : M' \to X'$. The goal of this section is to study the following problem: Given $f \in X'$ and $g \in M$,

$$\begin{cases} \text{Seek } u \in X \text{ and } p \in M' \text{ such that} \\ Au + B^T p = f, \\ Bu = g. \end{cases} \tag{A.19}$$

Denote by $\text{Ker}(B)$ the kernel of the operator B and define the operator $\pi A : \text{Ker}(B) \to \text{Ker}(B)'$ such that $\langle \pi A u, v\rangle = \langle Au, v\rangle$ for all $u, v \in \text{Ker}(B)$.

Theorem A.56. *Problem* (A.19) *is well-posed if and only if:*

(i) $\pi A : \mathrm{Ker}(B) \to \mathrm{Ker}(B)'$ *is an isomorphism.*
(ii) $B : X \to M$ *is surjective.*

Proof. (1) Let us first show that if problem (A.19) is well-posed, then statements (i) and (ii) are satisfied.

1.a Let h be in M, denote by (u,p) the solution to problem (A.19) with $f = 0$ and $g = h$. It is clear that u satisfies $Bu = h$. Hence, B is surjective.
1.b Let us show that πA is surjective. Let $h \in \mathrm{Ker}(B)'$. Owing to the Hahn–Banach Theorem, there is an extension $\tilde{h} \in X'$ such that $\langle \tilde{h}, v\rangle = \langle h, v\rangle$ for all v in $\mathrm{Ker}(B)$ and $\|\tilde{h}\|_{X'} = \|h\|_{\mathrm{Ker}(B)'}$. Let (u,p) be the solution to (A.19) with $f = \tilde{h}$ and $g = 0$. It is clear that u is in $\mathrm{Ker}(B)$. Since $\langle B^T p, v\rangle = \langle p, Bv\rangle = 0$ for all v in $\mathrm{Ker}(B)$, we infer $\langle \pi A u, v\rangle = \langle Au, v\rangle = \langle \tilde{h}, v\rangle = \langle h, v\rangle$ for all v in $\mathrm{Ker}(B)$. As a result, $\pi A u = h$.
1.c Let us show that πA is injective. Let u in $\mathrm{Ker}(B)$ be such that $\pi A u = 0$. Then, $\langle Au, v\rangle = 0$ for all v in $\mathrm{Ker}(B)$; as a result, Au is in $\mathrm{Ker}(B)^\perp$. B being surjective, $\mathrm{Im}(B)$ is closed and owing to Banach's Theorem, $\mathrm{Im}(B^T) = \mathrm{Ker}(B)^\perp$. As a result, $Au \in \mathrm{Im}(B^T)$, i.e., there is $p \in M'$ such that $Au = -B^T p$. Hence, $Au + B^T p = 0$ and $Bu = 0$, which shows that (u,p) is the solution to problem (A.19) with $f = 0$ and $g = 0$. The solution to (A.19) being unique, we infer $u = 0$.

(2) Conversely, assume that statements (i) and (ii) are satisfied.

2.a For $f \in X'$ and $g \in M$, let us show that there is at least one solution to problem (A.19). B being surjective, there is u_g in X such that $Bu_g = g$. The linear form $f - Au_g$ is continuous on X and $\mathrm{Ker}(B)$. Denote by $h_{f,g}$ the linear form on $\mathrm{Ker}(B)$ such that $\langle h_{f,g}, v\rangle = \langle f, v\rangle - \langle Au_g, v\rangle$ for all v in $\mathrm{Ker}(B)$. Let $\phi \in \mathrm{Ker}(B)$ be the solution to the problem $\pi A \phi = h_{f,g}$ and set $u = \phi + u_g$. The linear form $f - Au$ is clearly in $\mathrm{Ker}(B)^\perp$. Since B is surjective, $\mathrm{Ker}(B)^\perp = \mathrm{Im}(B^T)$; that is, there is $p \in M'$ such that $B^T p = f - Au$. Moreover, $Bu = B(\phi + u_g) = Bu_g = g$. Hence, we have constructed a solution to problem (A.19).
2.b Let us show that the solution is unique. Let (u,p) be such that $Bu = 0$ and $Au + B^T p = 0$. Clearly, $u \in \mathrm{Ker}(B)$ and $\pi A u = 0$. Since πA is injective, $u = 0$. As a result $B^T p = 0$. Since B is surjective, B^T is necessarily injective, which implies $p = 0$.

□

Remark A.57. Problem (A.19) is studied in detail in §**??** and §**??**. This problem can be interpreted as a saddle-point problem when the operator A is self-adjoint and monotone; see §**??**. □

B

Functional Analysis

This appendix collects the functional analysis concepts used in this book. We first restate fundamental results on Lebesgue integration together with the main properties of the $L^p(\Omega)$ spaces. Then, we introduce the concept of distributions and distributional derivatives. We review the main properties of the Sobolev spaces $W^{s,p}(\Omega)$. Most of the results are stated without proof, the reader being referred to specialized textbooks for complements [Ada75, Bar01, Bre91, MaZ97, Rud87, Sob63, Yos80].

Henceforth, Ω is a (measurable) open set of $\mathbb{R}^d$ with boundary $\partial\Omega$. Whenever it is well-defined, its outward normal is denoted by n.

B.1 Lebesgue and Lipschitz Spaces

B.1.1 Measurable functions and the Lebesgue integral

Let $\mathbb{M}(\Omega)$ be the space of scalar-valued functions on Ω that are *Lebesgue-measurable*. In particular, $\mathbb{M}(\Omega)$ contains functions that are piecewise continuous and, more generally, all the functions that are integrable in the Riemann sense. All the functions used in this book are measurable.

Measurable functions are defined up to sets of zero measure. In other words, $\mathbb{M}(\Omega)$ is a space of equivalence classes of functions, that is, two functions belong to the same equivalence class if they coincide *almost everywhere* (henceforth, a.e.), i.e., everywhere but on a set of zero Lebesgue measure.

Example B.1. The function $\phi :]0,1[\to \{0,1\}$ which takes the value 1 on rational numbers and which is zero otherwise belongs to the same equivalence class as the zero function. Hence, $\phi = 0$ a.e. on $]0,1[$. □

B.1.2 Lebesgue spaces

Definition B.2. $L^1(\Omega)$ *is the space of the scalar-valued functions that are Lebesgue-integrable. The space of* locally integrable *functions is denoted by*

$L^1_{\text{loc}}(\Omega)$ *and is defined as*

$$L^1_{\text{loc}}(\Omega) = \{f \in \mathbb{M}(\Omega);\, \forall \text{compact } K \subset \Omega,\ f \in L^1(K)\}. \tag{B.1}$$

Throughout this book, integrals are always understood in the Lebesgue sense. Whenever no confusion may arise and to alleviate the notation, we omit the Lebesgue measure under the integral sign; hence, we write $\int_\Omega f$ instead of $\int_\Omega f(x)\,\mathrm{d}x$.

Theorem B.3 (Lebesgue's Dominated Convergence). *Let $\{f_n\}_{n\geq 0}$ be a sequence of functions in $L^1(\Omega)$ such that:*

(i) *$f_n(x) \to f(x)$ a.e. in Ω.*
(ii) *There is $g \in L^1(\Omega)$ such that, for all n, $|f_n(x)| \leq g(x)$ a.e. in Ω.*

Then, $f \in L^1(\Omega)$ and $f_n \to f$ in $L^1(\Omega)$.

Proof. See [Bar01, p. 123], [Bre91, p. 54], or [Rud66]. □

Definition B.4. *For $1 \leq p \leq +\infty$, let*

$$L^p(\Omega) = \{f \in \mathbb{M}(\Omega);\, \|f\|_{0,p,\Omega} < +\infty\}, \tag{B.2}$$

where

$$\|f\|_{0,p,\Omega} = \left(\int_\Omega |f|^p\right)^{\frac{1}{p}}, \quad \textit{for } 1 \leq p < +\infty, \tag{B.3}$$

$$\|f\|_{0,\infty,\Omega} = \operatorname*{ess\,sup}_{x\in\Omega} |f(x)| = \inf\{M \geq 0;\, |f(x)| \leq M \textit{ a.e. on } \Omega\}. \tag{B.4}$$

Theorem B.5 (Fischer–Riesz). *Let $1 \leq p \leq +\infty$. Equipped with the $\|\cdot\|_{0,p,\Omega}$ norm, the vector space $L^p(\Omega)$ is a Banach space.*

Proof. See [Bar01, p. 142], [Sob63, §I.2.2], [Bre91, p. 57], or [Rud66]. □

In this book, we also employ the notation $\|f\|_{L^p(\Omega)} = \|f\|_{0,p,\Omega}$. For $1 \leq p \leq +\infty$, we denote by p' its *conjugate*, i.e., $\frac{1}{p} + \frac{1}{p'} = 1$ with the convention that $p' = 1$ if $p = +\infty$ and $p' = +\infty$ if $p = 1$.

Theorem B.6 (Hölder). *Let $f \in L^p(\Omega)$ and $g \in L^{p'}(\Omega)$ with $1 \leq p \leq +\infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$. Then, $fg \in L^1(\Omega)$ and*

$$\int_\Omega |fg| \leq \|f\|_{L^p(\Omega)} \|g\|_{L^{p'}(\Omega)}. \tag{B.5}$$

Proof. See [Bar01, p. 404], [Bre91, p. 56], or [Rud66]. □

The following corollary is an easy consequence of Hölder's inequality:

Corollary B.7 (Interpolation inequality). *Let $1 \leq p \leq q \leq +\infty$ and $0 \leq \alpha \leq 1$. Let r be such that $\frac{1}{r} = \frac{\alpha}{p} + \frac{1-\alpha}{q}$. Then,*

$$\forall f \in L^p(\Omega) \cap L^q(\Omega), \quad \|f\|_{L^r(\Omega)} \leq \|f\|^{\alpha}_{L^p(\Omega)} \|f\|^{1-\alpha}_{L^q(\Omega)}. \tag{B.6}$$

Theorem B.8 (Riesz' Representation Theorem). *Let* $1 \le p < +\infty$. *The dual space of* $L^p(\Omega)$ *can be identified with* $L^{p'}(\Omega)$.

Proof. See [Sob63, §I.3.3], [Bre91, p. 61], or [Rud66]. □

One important consequence of Theorem B.8 is that $L^p(\Omega)$ is reflexive if $1 < p < +\infty$. However, $L^1(\Omega)$ and $L^\infty(\Omega)$ are not reflexive. The dual of $L^1(\Omega)$ is $L^\infty(\Omega)$, but the dual of $L^\infty(\Omega)$ strictly contains $L^1(\Omega)$; see [Rud66] or [Bre91, p. 65]. Among all the Lebesgue spaces, $L^2(\Omega)$ plays a particular role owing to the following important consequence of the Fischer–Riesz Theorem:

Theorem B.9. $L^2(\Omega)$ *is a Hilbert space when equipped with the scalar product*

$$(f,g)_{L^2(\Omega)} = \int_\Omega fg.$$

Henceforth, we denote by $(\cdot\,,\cdot)_{0,\Omega}$ the scalar product in $L^2(\Omega)$, and when no confusion is possible, we denote the corresponding norm by

$$\|f\|_{0,\Omega} = \left(\int_\Omega f^2\right)^{\frac{1}{2}}.$$

In $L^2(\Omega)$, the Hölder inequality becomes the *Cauchy–Schwarz inequality*:

$$\forall f, g \in L^2(\Omega), \quad (f,g)_{0,\Omega} \le \|f\|_{0,\Omega}\|g\|_{0,\Omega}.$$

B.1.3 Spaces of continuous functions

Definition B.10 (Hölder spaces).

(i) *Let* E *be a subset of* $\overline{\Omega}$. $\mathcal{C}^0(E)$ *denotes the space of functions that are continuous on* E, *and for every integer* $k \ge 1$, $\mathcal{C}^k(E)$ *is the space of functions that are* k*-times continuously Fréchet-differentiable on* E.

(ii) *For* $0 < \alpha \le 1$, $\mathcal{C}^{0,\alpha}(E)$ *is the space of functions that are Hölder of exponent* α *on* E, *i.e.,* $f \in \mathcal{C}^{0,\alpha}(E)$ *if* f *is continuous on* E *and*

$$\sup_{x,y\in E} \frac{|f(x)-f(y)|}{|x-y|^\alpha} < +\infty.$$

For every integer $k \ge 1$, $\mathcal{C}^{k,\alpha}(E)$ *is the space of the functions* f *in* $\mathcal{C}^k(E)$ *such that* $\partial^\gamma f \in \mathcal{C}^{0,\alpha}(E)$ *for all multi-index* γ *of length* $|\gamma| = k$.

(iii) *For* $E = \overline{\Omega}$ *(resp.,* $E = \Omega$*),* $\mathcal{C}^{0,1}(E)$ *is called the space of globally (resp., locally)* Lipschitz functions.

Proposition B.11. *Let* $k \ge 0$ *be an integer and* $0 < \alpha \le 1$. *If* Ω *is bounded,* $\mathcal{C}^{k,\alpha}(\overline{\Omega})$ *is a Banach space when equipped with the norm*

$$\|f\|_{\mathcal{C}^{k,\alpha}(\overline{\Omega})} = \max_{|\gamma|\le k} \sup_{x\in\overline{\Omega}} |\partial^\gamma f(x)| + \max_{|\gamma|=k} \sup_{x,y\in\overline{\Omega}} \frac{|\partial^\gamma f(x) - \partial^\gamma f(y)|}{|x-y|^\alpha}.$$

Remark B.12. The above definitions extend to functions that are defined only at the boundary $\partial\Omega$. In particular, we denote by $\mathcal{C}^{0,1}(\partial\Omega)$ the space of Lipschitzian functions on $\partial\Omega$. □

B.2 Distributions

B.2.1 Preliminary definitions

Definition B.13. *$\mathcal{D}(\Omega)$ is the vector space of $\mathcal{C}^\infty$ functions whose support in Ω is compact.*

Theorem B.14. *Let $1 \le p < +\infty$. Then, $\mathcal{D}(\Omega)$ is dense in $L^p(\Omega)$.*

Proof. See [MaZ97, p. 5], [Bre91, p. 61], or [Rud66]. □

Lemma B.15. *Let $f \in L^1_{\text{loc}}(\Omega)$ be such that $\int_\Omega f\varphi = 0$, $\forall \varphi \in \mathcal{D}(\Omega)$. Then, $f = 0$ a.e. in Ω.*

Proof. See [MaZ97, p. 6], [Bre91, p. 61], or [Rud66]. □

Definition B.16 (Distributions). *A linear mapping*

$$u : \mathcal{D}(\Omega) \ni \varphi \longmapsto \langle u, \varphi \rangle_{\mathcal{D}',\mathcal{D}} \in \mathbb{R} \text{ or } \mathbb{C},$$

is said to be a distribution *on Ω if and only if the following property holds: For all compact K in Ω, there is an integer p and a constant c such that*

$$\forall \varphi \in \mathcal{D}(\Omega),\ \text{supp}(\varphi) \subset K, \quad |\langle u, \varphi \rangle_{\mathcal{D}',\mathcal{D}}| \le c \sup_{x\in K,\, |\alpha| \le p} |\partial^\alpha \varphi(x)|.$$

Remark B.17. $\mathcal{D}(\Omega)$ is not a normed vector space. As a result, $\mathcal{D}'(\Omega)$ is not the dual of $\mathcal{D}(\Omega)$ in the sense of Definition A.14. We shall nevertheless use the notation $\mathcal{D}'(\Omega)$. □

Example B.18.

(i) Every function f in $L^1_{\text{loc}}(\Omega)$ can be identified with the distribution

$$\tilde{f} : \mathcal{D}(\Omega) \ni \varphi \longmapsto \langle \tilde{f}, \varphi \rangle_{\mathcal{D}',\mathcal{D}} = \int_\Omega f\varphi.$$

This identification is possible owing to Lemma B.15 since, for $f, g \in L^1_{\text{loc}}(\Omega)$,

$$(f = g \quad \text{a.e. in } \Omega) \iff \left(\forall \varphi \in \mathcal{D}(\Omega),\ \int_\Omega f\varphi = \int_\Omega g\varphi\right).$$

We will constantly abuse the notation by identifying $f \in L^1_{\text{loc}}(\Omega)$ with the associated distribution $\tilde{f} \in \mathcal{D}'(\Omega)$.

(ii) Every function of $\mathcal{C}^{k,\alpha}(\Omega)$, $k \ge 0$, can be identified with a distribution.

(iii) Let a be a point in Ω. The *Dirac measure* at a is the distribution

$$\delta_{x=a} : \mathcal{D}(\Omega) \ni \varphi \longmapsto \langle \delta_{x=a}, \varphi \rangle_{\mathcal{D}',\mathcal{D}} = \varphi(a).$$

Note that $\delta_{x=a} \notin L^1(\Omega)$, i.e., there is no function $f \in L^1(\Omega)$ such that $\varphi(a) = \int_\Omega f\varphi$, $\forall \varphi \in \mathcal{D}(\Omega)$. Otherwise, one would have $0 = \int_\Omega f\varphi$ for all functions $\varphi \in \mathcal{D}(\Omega \backslash \{a\})$, which, owing to Lemma B.15, would imply $f = 0$ a.e. in $\Omega \backslash \{a\}$, i.e., $f = 0$ a.e. in Ω. □

B.2.2 The distributional derivative

The key to the distribution theory is that every distribution is differentiable in the following sense:

Definition B.19. *Let $u \in \mathcal{D}'(\Omega)$ be a distribution and let $1 \le i \le d$. The* distributional derivative $\partial_i u \in \mathcal{D}'(\Omega)$ *is defined as follows:*

$$\partial_i u : \mathcal{D}(\Omega) \ni \varphi \longmapsto \langle \partial_i u, \varphi \rangle_{\mathcal{D}',\mathcal{D}} = -\langle u, \partial_i \varphi \rangle_{\mathcal{D}',\mathcal{D}}.$$

More generally, for a multi-index α, the distribution $\partial^\alpha u$ is defined such that

$$\partial^\alpha u : \mathcal{D}(\Omega) \ni \varphi \longmapsto \langle \partial^\alpha u, \varphi \rangle_{\mathcal{D}',\mathcal{D}} = (-1)^{|\alpha|} \langle u, \partial^\alpha \varphi \rangle_{\mathcal{D}',\mathcal{D}}.$$

Hereafter, we set conventionally $\partial^0 u = u$ and $\nabla u = (\partial_1 u, \ldots, \partial_d u)$.

Remark B.20. The notion of distributional derivative is the extension of that of pointwise (or classical) derivative. This notion allows for the differentiation of functions that are not derivable in the classical sense. When $u \in L^1_{\mathrm{loc}}(\Omega)$, distributional derivatives of u are sometimes called *weak derivatives.* □

Proposition B.21. *Let α be a multi-index and let $u \in \mathcal{C}^{|\alpha|}(\Omega)$. Then, up to the order $|\alpha|$, the weak derivatives and the classical derivatives of u coincide.*

Proof. Let $(\partial^\alpha u)_{cl}$ denote the pointwise derivative; then,

$$\begin{aligned} \forall \varphi \in \mathcal{D}(\Omega), \quad \langle (\partial^\alpha u)_{cl}, \varphi \rangle_{\mathcal{D}',\mathcal{D}} &= \int_\Omega (\partial^\alpha u)_{cl}\, \varphi \quad (\text{since } (\partial^\alpha u)_{cl} \in L^1_{\mathrm{loc}}(\Omega)) \\ &= (-1)^{|\alpha|} \int_\Omega u\, \partial^\alpha \varphi \quad (\text{integration by parts}) \\ &= (-1)^{|\alpha|} \langle u, \partial^\alpha \varphi \rangle_{\mathcal{D}',\mathcal{D}} \quad (\text{since } u \in L^1_{\mathrm{loc}}(\Omega)). \end{aligned}$$

Note that in the integration by parts there are no boundary terms since $\partial^\beta \varphi_{|\partial\Omega} = 0$ for all multi-index β. □

Theorem B.22 (Rademacher). *Let $u \in \mathcal{C}^{0,1}(\Omega)$ be a locally Lipschitz function on Ω. Then, u is differentiable a.e. in Ω and the pointwise derivative coincides a.e. with the weak derivative.*

Proof. See [MaZ97, p. 44]. □

Example B.23.

(i) Take $\Omega =]-1, 1[$ and $u = 1 - |x|$. The weak derivative of u is

$$\partial_x u = \begin{cases} 1 & \text{if } x < 0, \\ -1 & \text{if } x > 0. \end{cases}$$

This example is fundamental. Its generalization to higher space dimension is used repeatedly in the book.

(ii) The *Heavyside function* (or step function) on $\Omega =]-1, 1[$ is defined by

$$H(x) = \begin{cases} 0 & \text{if } x < 0, \\ 1 & \text{if } x \geq 0. \end{cases}$$

It is clear that H is not derivable in the classical sense. However, a simple computation shows that $\partial_x H = \delta_{x=0}$. □

Lemma B.24. *Let $\Omega =]0,1[$. For all f in $L^1(\Omega)$,*

$$\partial_x \left(\int_0^x f \right) = f \qquad \text{in } \mathcal{D}'(\Omega). \tag{B.7}$$

Proof. We use a density argument. Since $\mathcal{D}(\Omega)$ is dense in $L^1(\Omega)$, there is a sequence $\{f_n\}_{n\geq 0}$ in $\mathcal{D}(\Omega)$ that converges to f in $L^1(\Omega)$ and such that $\|f_n\|_{1,\Omega} \leq 2\|f\|_{1,\Omega}$. For all ϕ in $\mathcal{D}(\Omega)$, it is clear that $|\int_0^1 f_n\phi - \int_0^1 f\phi| \leq (\sup_{x\in\Omega} |\phi(x)|) \int_0^1 |f_n - f| \to 0$. Likewise, $(\int_0^x f_n)\phi'(x) \to (\int_0^x f)\phi'(x)$ a.e. in Ω, and $|(\int_0^x f_n)\phi'(x)| \leq 2|\phi'(x)| \int_0^1 |f|$. Lebesgue's Dominated Convergence Theorem implies that $\int_0^1 (\int_0^x f_n)\phi'(x)\,\mathrm{d}x \to \int_0^1 (\int_0^x f)\phi'(x)\,\mathrm{d}x$. Passing to the limit in the relation

$$\int_0^1 \left(\int_0^x f_n \right) \phi'(x)\,\mathrm{d}x = -\int_0^1 f_n(x)\phi(x)\,\mathrm{d}x,$$

yields

$$\forall \phi \in \mathcal{D}(\Omega), \qquad \int_0^1 \left(\int_0^x f \right) \phi'(x)\,\mathrm{d}x = -\int_0^1 f(x)\phi(x)\,\mathrm{d}x.$$

This shows that (B.7) holds in the distribution sense. □

B.3 Sobolev Spaces

B.3.1 The $W^{s,p}(\Omega)$ spaces

Definition B.25 (Sobolev spaces). *Let s and p be two integers with $s \geq 0$ and $1 \leq p \leq +\infty$. The so-called* Sobolev space *$W^{s,p}(\Omega)$ is defined as*

$$W^{s,p}(\Omega) = \{u \in \mathcal{D}'(\Omega);\, \partial^\alpha u \in L^p(\Omega),\, |\alpha| \leq s\}, \tag{B.8}$$

where the derivatives are understood in the distribution sense.

Proposition B.26. *$W^{s,p}(\Omega)$ is a Banach space when equipped with the norm*

$$\|u\|_{W^{s,p}(\Omega)} = \sum_{|\alpha|\leq s} \|\partial^\alpha u\|_{L^p(\Omega)}. \tag{B.9}$$

The case $p = 2$ is particularly interesting since the spaces $W^{s,2}(\Omega)$ have a Hilbert structure, and we henceforth denote them by $H^s(\Omega)$.

Theorem B.27 (Hilbert Sobolev spaces). *Let $s \geq 0$. The space $H^s(\Omega) = W^{s,2}(\Omega)$ is a Hilbert space when equipped with the scalar product*

$$(u, v)_{s,\Omega} = \sum_{|\alpha| \leq s} \int_\Omega \partial^\alpha u \, \partial^\alpha v.$$

The associated norm is denoted by $\| \cdot \|_{s,\Omega}$.

Sometimes we shall also make use of the following notation:

$$\|v\|_{s,p,\Omega} = \|v\|_{W^{s,p}(\Omega)}, \qquad |v|^2_{s,p,\Omega} = \sum_{|\alpha|=s} \|D^\alpha v\|^2_{L^p(\Omega)}, \tag{B.10}$$

$$\|v\|_{s,\Omega} = \|v\|_{H^s(\Omega)}, \qquad |v|^2_{s,\Omega} = \sum_{|\alpha|=s} \|D^\alpha v\|^2_{L^2(\Omega)}. \tag{B.11}$$

Example B.28.

(i) $H^1(\Omega) = \{u \in L^2(\Omega); \partial_i u \in L^2(\Omega), 1 \leq i \leq d\}$ is a Hilbert space for the scalar product

$$(u, v)_{1,\Omega} = \int_\Omega uv + \int_\Omega \nabla u \cdot \nabla v = \int_\Omega uv + \sum_{i=1}^d \int_\Omega \partial_i u \, \partial_i v.$$

The associated norm is

$$\|u\|^2_{1,\Omega} = \int_\Omega u^2 + \int_\Omega (\nabla u)^2 = \|u\|^2_{0,\Omega} + |u|^2_{1,\Omega}.$$

(ii) Let $\Omega =]0, 1[$ and consider the function $u(x) = x^\alpha$ where $\alpha \in \mathbb{R}$. One easily verifies that $u \in L^2(\Omega)$ if $\alpha > -\frac{1}{2}$, $u \in H^1(\Omega)$ if $\alpha > \frac{1}{2}$, and, more generally $u \in H^s(\Omega)$ if $\alpha > s - \frac{1}{2}$.

(iii) The function shown in the left panel of Figure B.1 belongs to $L^2(\Omega)$ but not to $\mathcal{C}^0(\overline{\Omega})$. This function does not belong to $H^1(\Omega)$ either, since its distributional first derivative is the sum of two Dirac measures supported at the two discontinuity points. The function shown in the right panel of Figure B.1 belongs to $H^1(\Omega)$. However, it is neither in $\mathcal{C}^1(\overline{\Omega})$ nor in $H^2(\Omega)$ since its first derivative is discontinuous.

(iv) Let $\Omega = B(0, \frac{1}{2}) \subset \mathbb{R}^2$ be the ball centered at 0 and of radius $\frac{1}{2}$. A simple computation shows that the function

$$u(x_1, x_2) = \log\big(-\log(x_1^2 + x_2^2)\big)$$

is in $H^1(\Omega)$. This example shows that in two dimensions, functions in $H^1(\Omega)$ are neither necessarily continuous nor bounded. □

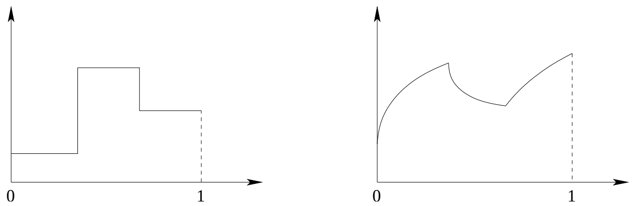

Fig. B.1. The function on the left is not $H^1(]0,1[)$, that on the right is.

The following lemma characterizes the nullspace of ∇; see [MaZ97, p. 24].

Lemma B.29. *Assume that Ω is an open connected set. Let $1 \leq p \leq +\infty$. Let u in $W^{1,p}(\Omega)$ such that $\nabla u = 0$ a.e. on Ω; then, u is constant.*

Definition B.30 (Fractional Sobolev spaces). *For $0 < s < 1$ and $1 \leq p < +\infty$, the so-called* Sobolev space with fractional exponent *is defined as*

$$W^{s,p}(\Omega) = \left\{ u \in L^p(\Omega);\ \frac{u(x) - u(y)}{\|x - y\|^{s+\frac{d}{p}}} \in L^p(\Omega \times \Omega) \right\}. \tag{B.12}$$

Furthermore, when $s > 1$ is not integer, letting $\sigma = s - [s]$ where $[s]$ is the integer part of s, $W^{s,p}(\Omega)$ is defined as

$$W^{s,p}(\Omega) = \{u \in W^{[s],p}(\Omega);\ \partial^\alpha u \in W^{\sigma,p}(\Omega),\ \forall \alpha,\ |\alpha| = [s]\}. \tag{B.13}$$

When $p = 2$, we denote $H^s(\Omega) = W^{s,2}(\Omega)$.

Remark B.31.

(i) The Sobolev spaces with fractional exponent can also be introduced by interpolation between $L^p(\Omega)$ and $W^{1,p}(\Omega)$ or by Fourier transform if $p = 2$ and Ω is either $\mathbb{R}^d$ or the d-torus; see [Ada75, LiM68].

(ii) Using mappings, it is possible to define $W^{s,p}(\partial\Omega)$ whenever $\partial\Omega$ is a smooth manifold. □

Example B.32.

(i) It can be easily verified that the function shown in the left panel of Figure B.1 is in $W^{s,p}(]0,1[)$ for all $0 < s$ and $1 \leq p < +\infty$ such that $sp < 1$.

(ii) One can also verify that if Ω is bounded and $1 \leq p < +\infty$, $\mathcal{C}^{0,\alpha}(\overline{\Omega}) \subset W^{s,p}(\Omega)$ provided $0 \leq s < \alpha \leq 1$. □

B.3.2 Density

As an alternative to Definition B.25, one can also define Sobolev spaces as follows: Let $X^{s,p}(\Omega)$ be the closure of $\mathcal{C}^s(\Omega) \cap W^{s,p}(\Omega)$ with respect to the Sobolev norm $\|\cdot\|_{W^{s,p}(\Omega)}$. Observe that $X^{s,\infty}(\Omega) \neq W^{s,\infty}(\Omega)$ since $X^{s,\infty}(\Omega) = \mathcal{C}^s_b(\Omega)$, where $\mathcal{C}^s_b(\Omega)$ is the Banach space composed of the functions whose derivatives up to order s are continuous and bounded on Ω. However, for $1 \leq p < +\infty$, $X^{s,p}(\Omega) = W^{s,p}(\Omega)$. More precisely, the following result is proved in [MeS64]:

Theorem B.33 (Meyers–Serrin). *Let Ω be any open set. Then $\mathcal{C}^\infty(\Omega) \cap W^{s,p}(\Omega)$ is dense in $W^{s,p}(\Omega)$ for $p < +\infty$.*

Theorem B.34 (Friedrichs). *Let Ω be any open set and let $u \in W^{1,p}(\Omega)$ with $1 \le p < +\infty$. Then, there exists a sequence $\{u_n\}_{n\ge 0}$ in $\mathcal{D}(\mathbb{R}^d)$ such that*

(i) *$u_n \to u$ in $L^p(\Omega)$.*
(ii) *$\nabla u_{n|\omega} \to \nabla u_{|\omega}$ in $[L^p(\omega)]^d$ for all ω such that $\overline{\omega} \subset \Omega$ and $\overline{\omega}$ compact.*

Proof. See [Bre91, p. 151]. □

Remark B.35. In general, $\mathcal{C}^\infty(\overline{\Omega})$ is not dense in $W^{s,p}(\Omega)$. For this stronger density result to hold, some additional regularity hypotheses on Ω must be assumed. In particular, this result cannot hold whenever Ω lies on both sides of parts of its boundary. For instance, think of the function $u(x_1, x_2) = r\cos(\frac{1}{2}\theta)$ in $\Omega = \{0 \le r < 1,\ 0 < \theta < 2\pi\}$, where (r, θ) are the cylindrical coordinates. Let B be the closed ball of radius 1 centered at 0. Note that $B = \overline{\Omega}$. It is clear that u is in $W^{1,p}(\Omega)$, but u is not in $W^{1,p}(B)$. Indeed, $\partial_\theta u = -\frac{1}{2}r\sin(\frac{1}{2}\theta) + 2r\delta_{\theta=0}$ where $\delta_{\theta=0}$ is the Dirac measure supported by the segment $\{0 < x_1 < 1,\ x_2 = 0\}$, and $\delta_{\theta=0}$ cannot be identified with any function in $L^p(\Omega)$; see Example B.18(iii). Since $\mathcal{C}^1(\overline{\Omega}) \subset W^{1,p}(B)$ and $W^{1,p}(B)$ is complete, the closure of $\mathcal{C}^1(\overline{\Omega})$ in $W^{1,p}(B)$ is a subspace of $W^{1,p}(B)$. Since $u \notin W^{1,p}(B)$, u cannot be a limit of functions in $\mathcal{C}^1(\overline{\Omega})$. □

To extend the previous density results, we introduce the following:

Definition B.36 ((s,p)-extension property). *Let $1 \le p \le +\infty$ and let $s \ge 0$. Ω is said to have the (s,p)-*extension property *if there is a bounded linear operator $L : W^{s,p}(\Omega) \to W^{s,p}(\mathbb{R}^d)$ such that $Lu_{|\Omega} = u$ for all $u \in W^{s,p}(\Omega)$. For $u \in W^{s,p}(\Omega)$, Lu is called (s,p)-*extension *of u.*

A fundamental result of Calderón–Stein [Cal61, Ste70] is the following:

Theorem B.37. *Every open set with a Lipschitz boundary has the $(1,p)$-extension property.*

Corollary B.38. *Let Ω be a bounded open set having the $(1,p)$-extension property. The restriction to Ω of functions in $\mathcal{D}(\mathbb{R}^d)$ span a dense subspace of $W^{1,p}(\Omega)$.*

Proof. Apply Theorem B.34 to a $(1,p)$-extension of $u \in W^{1,p}(\Omega)$. □

B.3.3 Embedding and compacity

Proposition B.39. *Let Ω be an open bounded set. Then, for $1 \le p < q \le +\infty$, the embedding $L^q(\Omega) \subset L^p(\Omega)$ is continuous.*

Proof. Easy consequence of Hölder's inequality. □

One of the key arguments in the embedding theory is the following:

Theorem B.40 (Sobolev). *Let* $1 \le p < d$ *and denote by* p^* *the number such that* $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{d}$. *Then,*

$$\exists c = \frac{p^*}{1^*},\ \forall u \in W^{1,p}(\mathbb{R}^d), \quad \|u\|_{L^{p^*}(\mathbb{R}^d)} \le c\,\|\nabla u\|_{L^p(\mathbb{R}^d)}. \tag{B.14}$$

Proof. See [MaZ97, p. 32], [Sob63, §I.7.4], or [Bre91, p. 162]. □

Corollary B.41. *Let* $1 \le p, q \le +\infty$. *The following embeddings are continuous:*

$$W^{1,p}(\mathbb{R}^d) \subset L^q(\mathbb{R}^d) \text{ if } \begin{cases} \text{either } 1 \le p < d \text{ and } p \le q \le p^*, \\ \text{or } p = d \text{ and } p \le q < +\infty. \end{cases} \tag{B.15}$$

Proof. See [MaZ97, p. 34], [Sob63, §I.8.2], or [Bre91, p. 165]. □

Theorem B.42 (Morrey). *Let* $d < p \le +\infty$ *and* $\alpha = 1 - \frac{d}{p}$. *The following embedding is continuous:*

$$W^{1,p}(\mathbb{R}^d) \subset L^\infty(\mathbb{R}^d) \cap \mathcal{C}^{0,\alpha}(\mathbb{R}^d). \tag{B.16}$$

Proof. See [MaZ97, p. 37] or [Bre91, p. 166]. □

Corollary B.43. *Let* $1 \le p, q \le +\infty$. *Let* $s \ge 1$ *be an integer. Let* Ω *be a bounded open set having the* $(1,p)$*-extension property. The following embeddings are continuous:*

$$W^{s,p}(\Omega) \subset \begin{cases} L^q(\Omega) & \text{if } 1 \le p < \frac{d}{s} \text{ and } p \le q \le p^*, \\ L^q(\Omega) & \text{if } p = \frac{d}{s} \text{ and } p \le q < +\infty, \\ L^\infty(\Omega) \cap \mathcal{C}^{0,\alpha}(\overline{\Omega}) & \text{if } p > \frac{d}{s} \text{ and } \alpha = 1 - \frac{d}{sp}. \end{cases} \tag{B.17}$$

Remark B.44. This theorem implies that in one dimension, the functions of $H^1(\Omega)$ are continuous, whereas in dimension 2 or 3, this may not be the case; see Example B.28(iv). If $d = 2$ or 3, functions in $H^2(\Omega)$ are continuous. □

Example B.45. Let $1 < \alpha$, $1 \le p < 2$, and $\Omega = \{(x_1, x_2) \in \mathbb{R}^2;\ 0 < x_1 < 1,\ 0 < x_2 < x_1^\alpha\}$. Let $u(x_1, x_2) = x_1^\beta$ with $1 - \frac{1+\alpha}{p} < \beta < 0$. Then, $u \in W^{1,p}(\Omega)$ and $u \in L^q(\Omega)$ for all q such that $1 \le q < p_\alpha$ where $\frac{1}{p_\alpha} = \frac{1}{p} - \frac{1}{1+\alpha}$. Set $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{2}$ and $\epsilon = \frac{\beta-1}{1+\alpha} + \frac{1}{p} > 0$; one can choose β so that ϵ is arbitrarily small. Set $\frac{1}{p_\beta} = \frac{1}{p_\alpha} - \epsilon$. If ϵ is small enough, $p_\alpha < p_\beta < p^*$. Then $u \notin L^q(\Omega)$ for $p_\beta \le q \le p^*$, which would contradict Corollary B.43 if the $(1,p)$-extension hypothesis had been omitted. The hypothesis $1 < \alpha$ means that Ω has a cusp at the origin; hence, Ω is not Lipschitz. This counterexample shows that some regularity on Ω is needed for Corollary B.43 to hold. □

We conclude this section by stating a very useful compacity result.

Theorem B.46 (Rellich–Kondrachov). *Let $1 \leq p \leq +\infty$ and let $s \geq 0$. Let Ω be a bounded open set having the (s,p)-extension property. The following injections are compact:*

(i) *If $sp \leq d$, $W^{s,p}(\Omega) \subset L^q(\Omega)$ for all $1 \leq q < p^*$ where $\frac{1}{p^*} = \frac{1}{p} - \frac{s}{d}$.*
(ii) *If $sp > d$, $W^{s,p}(\Omega) \subset \mathcal{C}^0(\overline{\Omega})$.*

Proof. See [MaZ97], [BrS94, Chap. 1], or [Bre91, Chap. 8]. □

B.3.4 $W_0^{s,p}(\Omega)$ and its dual

Definition B.47. *For $1 \leq p < +\infty$ and $s \geq 0$, set*

$$W_0^{s,p}(\Omega) = \overline{\mathcal{D}(\Omega)}^{W^{s,p}(\Omega)}, \tag{B.18}$$

and let $W^{-s,p'}(\Omega) = \left(W_0^{s,p}(\Omega)\right)'$ be the dual of $W_0^{s,p}(\Omega)$ with the norm

$$\forall f \in W^{-s,p'}(\Omega), \qquad \|f\|_{W^{-s,p'}(\Omega)} = \sup_{u \in W_0^{s,p}(\Omega)} \frac{\langle f, u\rangle_{W^{-s,p'},W_0^{s,p}}}{\|u\|_{W^{s,p}(\Omega)}}.$$

For $p = 2$, set $H_0^s(\Omega) = W_0^{s,2}(\Omega)$ and $H^{-s}(\Omega) = (W_0^{s,2}(\Omega))'$.

Proposition B.48. *Let $1 \leq p < +\infty$ and let Ω be an open set. If u is in $W_0^{1,p}(\Omega)$, the zero-extension of u is in $W^{1,p}(\mathbb{R}^d)$.*

Corollary B.49. *The conclusions of Corollary* B.43 *and Theorem* B.46 *hold in $W_0^{1,p}(\Omega)$ without the $(1,p)$-extension hypothesis on Ω.*

The distributions in $W^{-1,p'}(\Omega)$ are characterized by the following:

Proposition B.50. *Let $1 \leq p < +\infty$, let Ω be an open set, and let $F \in W^{-1,p'}(\Omega)$. Then, there are $f_0, f_1, \ldots, f_d \in L^{p'}(\Omega)$ such that*

$$\forall v \in W_0^{1,p}(\Omega), \quad \langle F, v\rangle_{W^{-1,p'},W^{1,p}} = \int_\Omega f_0 v + \sum_{k=1}^{d} \int_\Omega f_k \partial_{x_k} v,$$

and $\|F\|_{W^{-1,p'}(\Omega)} = \max_{0\leq k\leq d} \|f_k\|_{L^{p'}(\Omega)}$. One can set $f_0 = 0$ if Ω is bounded.

Proof. See [Rud66] or [Bre91, p. 175]. □

Remark B.51.
(i) The notation for the dual of $W^{s,p}(\Omega)$ has been chosen so that when $s = 0$, it is coherent with the Riesz Theorem.
(ii) It is remarkable that some of the objects contained in $W^{-s,p'}(\Omega)$ are not functions but distributions. For instance, if $sp > d$, the Dirac measure at $x \in \Omega$ is in $W^{-s,p'}(\Omega)$. For a function u in $L^p(\Omega)$, all the first-order distributional derivatives of u are in $W^{-1,p'}(\Omega)$. □

B.3.5 The trace theory

In general, it is meaningless to speak of the value at $\partial\Omega$ of a function in $L^p(\Omega)$. For instance, take $\Omega =]0,1[^2$ and $u(x_1,x_2) = x_1^{-\frac{\alpha}{p}}$ with $0 < \alpha < 1$. It is clear that $u \in L^p(\Omega)$, but $u_{|x_1=0} = +\infty$. Let $\gamma_0 : \mathcal{C}^0(\overline{\Omega}) \to \mathcal{C}^0(\partial\Omega)$ map functions in $\mathcal{C}^0(\overline{\Omega})$ to their trace on $\partial\Omega$. If Ω is a Lipschitz bounded open set, γ_0 can be continuously extended to $W^{1,p}(\Omega)$; see, e.g., [Ada75]. Let us abuse the notation by still denoting the extension in question by γ_0.

Theorem B.52 (Trace Theorem 1). *Let $1 \le p < +\infty$ and Ω be a Lipschitz bounded open set. Then,*

(i) $\gamma_0 : W^{1,p}(\Omega) \to W^{\frac{1}{p'},p}(\partial\Omega)$ *is surjective.*
(ii) *The kernel of γ_0 is $W_0^{1,p}(\Omega)$.*

Statement (ii) in this theorem means that the functions in $W_0^{1,p}(\Omega)$ are those in $W^{1,p}(\Omega)$ that are zero at $\partial\Omega$.

When $p = 2$, we set $H^{\frac{1}{2}}(\partial\Omega) = W^{\frac{1}{2},2}(\partial\Omega)$. Statement (i) in Theorem B.52 implies that every function in $H^{\frac{1}{2}}(\partial\Omega)$ is the trace of a function in $H^1(\Omega)$. More precisely, from the Open Mapping Theorem we deduce the following:

Corollary B.53. *Let $1 \le p < +\infty$ and Ω be a Lipschitz bounded open set. Then, there exists a constant c such that, $\forall g \in W^{\frac{1}{p'},p}(\partial\Omega)$, there exists $u_g \in W^{1,p}(\Omega)$ satisfying*

$$\gamma_0(u_g) = g \qquad \textit{and} \qquad \|u\|_{W^{1,p}(\Omega)} \le c\,\|g\|_{W^{\frac{1}{p'},p}(\partial\Omega)}.$$

The function u_g is said to be a lifting *of g in $W^{1,p}(\Omega)$.*

Likewise $\partial_n u = n\cdot\nabla u$ is meaningful for functions in $W^{2,p}(\Omega)$ since, for such functions, $\nabla u \in W^{\frac{1}{p'},p}(\partial\Omega)$. Actually, provided Ω is smooth enough, the mapping $\gamma_1 : \mathcal{C}^1(\overline{\Omega}) \to \mathcal{C}^0(\partial\Omega)$ such that $\gamma_1(u) = \partial_n u$ can be continuously extended from $W^{2,p}(\Omega)$ to $W^{\frac{1}{p'},p}(\Omega)$.

Theorem B.54 (Trace Theorem 2). *Let $1 \le p < +\infty$ and Ω be a bounded open set of class $\mathcal{C}^2$. Then:*

(i) $(\gamma_0,\gamma_1) : W^{2,p}(\Omega)\times W^{2,p}(\Omega) \to W^{1+\frac{1}{p'},p}(\partial\Omega)\times W^{\frac{1}{p'},p}(\partial\Omega)$ *is surjective.*
(ii) *The kernel of (γ_0,γ_1) is $W_0^{2,p}(\Omega)$.*

Item (ii) in this theorem says that the functions in $W_0^{2,p}(\Omega)$ are those in $W^{2,p}(\Omega)$ that are zero at $\partial\Omega$ and whose normal derivative on $\partial\Omega$ is zero.

Remark B.55. If Ω is a polyhedron, the outward normal vector n is discontinuous at the edges. As a result, the normal derivative $\partial_n u$ cannot be smooth whatever the regularity of u. It is nevertheless possible to extend the results of Theorem B.54 to this situation [GiR86, p. 9]. For example, when Ω is a polygon in $\mathbb{R}^2$ with sides $\partial\Omega_j$ and normal vectors n_j, $1 \le j \le J$, the mapping $u \mapsto (\partial_{n_1}u,\ldots,\partial_{n_J}u)$ is continuous and surjective from $W^{2,p}(\Omega)$ to $\prod_{j=1}^{J} W^{\frac{1}{p'},p}(\partial\Omega_j)$. □

B.3.6 The divergence formula and its consequences

For brevity, we henceforth omit the symbols γ_0 and γ_1 to denote the trace of a distribution or that of its normal derivative. In this section, p is a number in $[1, +\infty[$ and Ω denotes a Lipschitz bounded open set.

Lemma B.56 (Divergence formula). *Let u be a smooth vector field. Then,*

$$\int_\Omega \nabla\cdot u = \int_{\partial\Omega} u\cdot n. \tag{B.19}$$

Corollary B.57. *Let $u \in [L^p(\Omega)]^d$ such that $\nabla\cdot u \in L^p(\Omega)$. Then, $u\cdot n \in W^{-\frac{1}{p},p}(\partial\Omega)$ and*

$$\forall q \in W^{1,p'}(\Omega), \qquad \int_\Omega q\nabla\cdot u = -\int_\Omega u\cdot\nabla q + \int_{\partial\Omega} q\, u\cdot n. \tag{B.20}$$

Proof. Use (B.19) for smooth functions, together with $\nabla\cdot(qu) = q\nabla\cdot u + u\cdot\nabla q$. Conclude using a density argument. □

Corollary B.58. *Let $u \in [L^p(\Omega)]^3$ such that $\nabla\times u \in [L^p(\Omega)]^3$. Then, $u \times n \in [W^{-\frac{1}{p},p}(\partial\Omega)]^3$ and*

$$\forall v \in W^{1,p'}(\Omega), \qquad \int_\Omega (\nabla\times u)\cdot v = \int_\Omega u\cdot(\nabla\times v) - \int_{\partial\Omega} (u \times n)\cdot v. \tag{B.21}$$

Proof. Use $\nabla\cdot(u\times v) = (\nabla\times u)\cdot v - u\cdot(\nabla\times v)$ together with (B.19) and a density argument. □

Corollary B.59 (Green's formula). *Let $\sigma \in [L^\infty(\Omega)]^{d\times d}$ and let $u \in W^{1,p}(\Omega)$ be such that $\nabla\cdot(\sigma\cdot\nabla u) \in L^p(\Omega)$. Then, $n\cdot\sigma\cdot\nabla u \in W^{-\frac{1}{p},p}(\partial\Omega)$ and*

$$\forall v \in W^{1,p'}(\Omega), \quad -\int_\Omega \nabla\cdot(\sigma\cdot\nabla u)\, v = \int_\Omega \nabla v\cdot\sigma\cdot\nabla u - \int_{\partial\Omega} (n\cdot\sigma\cdot\nabla u) v. \tag{B.22}$$

Proof. Use $\nabla\cdot((\sigma\cdot\nabla u)\, v) = \nabla v\cdot\sigma\cdot\nabla u + \nabla\cdot(\sigma\cdot\nabla u)\, v$ together with (B.19) and a density argument. □

Remark B.60. To be rigorous, the boundary integrals in (B.20), (B.21), and (B.22) should be written in the form of duality products. For instance, in (B.20), we should write $\langle u\cdot n, q\rangle_{W^{-\frac{1}{p},p}(\partial\Omega), W^{\frac{1}{p},p'}(\partial\Omega)}$. □

B.3.7 Poincaré-like inequalities

Lemma B.61 (Poincaré). *Let $1 \le p < +\infty$ and let Ω be a bounded open set. Then, there exists $c_{p,\Omega} > 0$ such that*

$$\forall v \in W_0^{1,p}(\Omega), \quad c_{p,\Omega}\|v\|_{L^p(\Omega)} \le \|\nabla v\|_{L^p(\Omega)}. \tag{B.23}$$

For $p = 2$, we denote $c_\Omega = c_{2,\Omega}$.

Proof. We only give the proof for $p < d$. Let $\tilde{v} \in W^{1,p}(\mathbb{R}^d)$ be the zero-extension of v; see Proposition B.48. Theorem B.40 implies $\|\tilde{v}\|_{L^{p^*}(\mathbb{R}^d)} \leq c\,\|\nabla\tilde{v}\|_{L^p(\mathbb{R}^d)}$. Since Ω is bounded and $p^* \geq p$, we infer $\|v\|_{L^p(\Omega)} = \|\tilde{v}\|_{L^p(\mathbb{R}^d)} \leq c\,\|\tilde{v}\|_{L^{p^*}(\mathbb{R}^d)}$, yielding (B.23). □

Remark B.62.

(i) The Poincaré inequality shows that on $W_0^{1,p}(\Omega)$, the seminorm $v \mapsto |v|_{1,p,\Omega}$ is equivalent to the usual norm $v \mapsto \|v\|_{1,p,\Omega}$. For instance, in the Hilbertian case ($p = 2$),

$$\forall v \in H_0^1(\Omega), \qquad \frac{c_\Omega}{(1+c_\Omega^2)^{\frac{1}{2}}}\|v\|_{1,\Omega} \leq |v|_{1,\Omega} \leq \|v\|_{1,\Omega}.$$

(ii) To give some further insight into (B.23), we give a proof in one dimension using $\Omega =]0,1[$. For $\varphi \in \mathcal{D}(\Omega)$, use $\varphi(0) = 0$ to write $\varphi(x) = \int_0^x \varphi'(y)\,\mathrm{d}y$ for all $x \in \Omega$. Hence,

$$\begin{aligned}\|\varphi\|_{0,p,\Omega}^p = \int_0^1 |\varphi(x)|^p\,\mathrm{d}x &= \int_0^1 \left|\int_0^x \varphi'(y)\,\mathrm{d}y\right|^p\,\mathrm{d}x \\ &\leq \int_0^1 \left(\int_0^x \mathrm{d}y\right)^{\frac{p}{p'}} \left(\int_0^x |\varphi'(y)|^p\,\mathrm{d}y\right)\,\mathrm{d}x \\ &\leq \int_0^1 |\varphi'(y)|^p\,\mathrm{d}y = |\varphi|_{1,p,\Omega}^p.\end{aligned}$$

Conclude using the density of $\mathcal{D}(\Omega)$ in $W_0^{1,p}(\Omega)$.

(iii) Let B_R be the ball with radius R. Using a scaling argument, one readily infers that $Rc_{p,B_R} = c_{p,B_1}$. □

Lemma B.63. *Let* $1 \leq p < +\infty$ *and* Ω *be a bounded connected open set having the* $(1,p)$*-extension property. Let* f *be a linear form on* $W^{1,p}(\Omega)$ *whose restriction on constant functions is not zero. Then, there is* $c_{p,\Omega} > 0$ *such that*

$$\forall v \in W^{1,p}(\Omega), \quad c_{p,\Omega}\|v\|_{W^{1,p}(\Omega)} \leq \|\nabla v\|_{L^p(\Omega)} + |f(v)|. \tag{B.24}$$

Proof. Use the Petree–Tartar Lemma. To this end, set $X = W^{1,p}(\Omega)$, $Y = [L^p(\Omega)]^d \times \mathbb{R}$, $Z = L^p(\Omega)$, and $A : X \ni v \mapsto (\nabla v, f(v)) \in Y$. Owing to Lemma B.29 and the hypotheses on f, A is continuous and injective. Moreover, the injection $X \subset Z$ is compact owing to Theorem B.46. □

Example B.64. Let E be a subset of Ω of non-zero measure and set $f(v) = \mathrm{meas}(E)^{-1}\int_E v$. It is clear that f is continuous, and if c is a constant function, $f(c)$ is zero if and only if c is zero. A second possibility consists of setting $f(v) = \mathrm{meas}(\partial\Omega_1)^{-1}\int_{\partial\Omega_1} v$ where $\partial\Omega_1$ is a subset of $\partial\Omega$ of non-zero $(d-1)$-measure. The continuity of f is a consequence of the Trace Theorem B.52. □

Corollary B.65. *Under the hypotheses of Lemma* B.63*, assume* $f(1_\Omega) = 1$*, where* 1_Ω *is the constant function equal to* 1 *on* Ω*. Then,*

$$\forall v \in W^{1,p}(\Omega), \quad c\,\|v - f(v)\|_{W^{1,p}(\Omega)} \leq \|\nabla v\|_{L^p(\Omega)}. \tag{B.25}$$

As an easy consequence of Lemma B.63, we infer a useful generalization of the Poincaré inequality.

Lemma B.66 (Poincaré–Friedrichs). *Under the above hypotheses, let* $W = \{v \in W^{1,p}(\Omega);\ f(v) = 0\}$. *Then,* W *is a closed subspace of* $W^{1,p}(\Omega)$ *and*

$$\forall v \in W, \quad c\,\|v\|_{W^{1,p}(\Omega)} \le \|\nabla v\|_{L^p(\Omega)}. \tag{B.26}$$

We conclude this section with two lemmas that play an important role in numerical analysis. In particular, they simplify considerably the error analysis of the finite element method. Recall that $\mathbb{P}_l$ is the vector space of real-valued polynomials in the variables $x_1, \ldots, x_d$ and of global degree at most l.

Lemma B.67 (Deny–Lions). *Let* $1 \le p \le +\infty$ *and let* Ω *be a connected bounded open set having the* $(1,p)$*-extension property. Let* $l \ge 0$. *There exists* $c > 0$ *such that*

$$\forall v \in W^{l+1,p}(\Omega), \qquad \inf_{\pi \in \mathbb{P}_l} \|v+\pi\|_{l+1,p,\Omega} \le c\,|v|_{l+1,p,\Omega}. \tag{B.27}$$

Proof. Let $N_l = \dim \mathbb{P}_l$, and introduce the N_l continuous linear forms

$$f_\alpha : W^{l+1,p}(\Omega) \ni v \longmapsto f_\alpha(v) = \int_\Omega \partial^\alpha v \in \mathbb{R}, \qquad |\alpha| \le l,$$

where $\alpha = (\alpha_1, \ldots, \alpha_d)$ is a multi-index. Define $X = W^{l+1,p}(\Omega)$, $Y = [L^p(\Omega)]^{N_{l+1}-N_l} \times \mathbb{R}^{N_l}$, and $Z = W^{l,p}(\Omega)$. Define the operator

$$A : X \ni v \longmapsto ((D^\alpha v)_{|\alpha|=l+1}, (f_\alpha(v))_{|\alpha|\le l}) \in Y.$$

A is clearly continuous. Let $v \in W^{l+1,p}(\Omega)$ and assume that $Av = 0$. Repeated application of Lemma B.29 yields $v \in \mathbb{P}_l$ and $f_\alpha(v) = 0$ for all $|\alpha| \le l$. Owing to the definition of the linear forms f_α, $v = 0$. That is to say, A is injective. Owing to the Rellich–Kondrachov Theorem, the injection $X = W^{l+1,p}(\Omega) \subset W^{l,p}(\Omega) = Z$ is compact. Then, the Petree–Tartar Lemma implies that there is $c > 0$ such that

$$\forall v \in W^{l+1,p}(\Omega), \qquad c\,\|v\|_{l+1,p,\Omega} \le |v|_{l+1,p,\Omega} + \sum_{|\alpha|\le l} |f_\alpha(v)|. \tag{B.28}$$

Let $\pi(v) \in \mathbb{P}_l$ be such that $f_\alpha(v + \pi(v)) = 0$ for $|\alpha| \le l$. Then,

$$\inf_{\pi \in \mathbb{P}_l} \|v+\pi\|_{l+1,p,\Omega} \le \|v+\pi(v)\|_{l+1,p,\Omega} \le c\,|v+\pi(v)|_{l+1,p,\Omega} = c\,|v|_{l+1,p,\Omega},$$

and this completes the proof; see [DeL55]. □

Lemma B.68 (Bramble–Hilbert). *Assume the hypotheses of the Deny–Lions Lemma hold. Then, there is* $c > 0$ *such that, for all* $f \in (W^{k+1,p}(\Omega))'$ *vanishing on* $\mathbb{P}_k$,

$$\forall v \in W^{k+1,p}(\Omega), \quad |f(v)| \le c\,\|f\|_{(W^{k+1,p}(\Omega))'}|v|_{k+1,p,\Omega}. \tag{B.29}$$

Proof. For all $\pi \in \mathbb{P}_k$, $|f(v)| = |f(v+\pi)| \le \|f\|_{(W^{k+1,p}(\Omega))'}\|v+\pi\|_{k+1,p,\Omega}$, that is, $|f(v)| \le \|f\|_{(W^{k+1,p}(\Omega))'} \inf_{\pi\in\mathbb{P}_k} \|v+\pi\|_{k+1,p,\Omega}$. Then, conclude using the Deny–Lions Lemma; see [BrH70]. □

B.3.8 The right inverse of the divergence operator

This section investigates the surjectivity of the divergence operator. An open bounded set Ω is said to be star-shaped with respect to a ball B if for any $x \in \Omega$ and $z \in B \subset \Omega$, the segment joining x and z is contained in Ω.

Lemma B.69 (Bogovskiĭ). *Assume $d \geq 2$ and let Ω be a bounded open set in $\mathbb{R}^d$ star-shaped with respect to a ball B. Let $1 < p < +\infty$ and set*

$$L^p_{\int=0}(\Omega) = \left\{ v \in L^p(\Omega);\ \int_\Omega v = 0 \right\}.$$

Then, the operator $\nabla\cdot : [W^{1,p}_0(\Omega)]^d \rightarrow L^p_{\int=0}(\Omega)$ is surjective.

Proof. See [Bog80], [Gal94, Lemma 3.1, Chap. III], or [DuM01]. □

Remark B.70. The hypothesis "Ω is star-shaped" can be replaced by "the boundary of Ω is Lipschitz" when $p = 2$; see [GiR86, pp. 18–26]. □

Owing to Lemma A.42 and Theorem B.8, Lemma B.69 implies:

Corollary B.71. *Under the hypotheses of Lemma* B.69*, there is $\beta > 0$ such that*

$$\inf_{q \in L^{p'}_0(\Omega)} \sup_{v \in [W^{1,p}_0(\Omega)]^d} \frac{\int_\Omega q \nabla\cdot v}{\|v\|_{[W^{1,p}(\Omega)]^d} \|q\|_{L^{p'}(\Omega)}} \geq \beta. \tag{B.30}$$

Remark B.72. Lemma B.69 has been extended to fractional Sobolev spaces $[W^{s,p}(\Omega)]^d$ by Solonnikov [Sol01, Prop. 2.1]. □

As a consequence of Lemma B.69, we deduce a "coarse" version of de Rham's Theorem in $L^p(\Omega)$.

Theorem B.73 (de Rham). *Under the hypotheses of Lemma* B.69*, the continuous linear forms on $[W^{1,p}_0(\Omega)]^d$ that are zero on* $\mathrm{Ker}(\nabla\cdot)$ *are gradients of functions in $L^{p'}_{\int=0}(\Omega)$.*

Proof. This a simple consequence of the Closed Range Theorem. Indeed, consider the weak gradient operator $\nabla : L^{p'}_{\int=0}(\Omega) \rightarrow [W^{-1,p'}(\Omega)]^d$. It is clear that $-\nabla = (\nabla\cdot)^T$. Since $\nabla\cdot$ is surjective by Lemma B.69, the Closed Range Theorem implies $[\mathrm{Ker}(\nabla\cdot)]^\perp = \mathrm{Im}(\nabla)$. □

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