AN INTRINSIC CRITERION FOR THE BIJECTIVITY OF HILBERT OPERATORS RELATED TO FRIEDRICHS' SYSTEMS

A. ERN[1], J.-L. GUERMOND[1,2], AND G. CAPLAIN[1]

Abstract. Friedrichs' theory of symmetric positive systems of first-order PDE's is revisited so as to avoid invoking traces at the boundary. Two intrinsic geometric conditions are introduced to characterize admissible boundary conditions. It is shown that the space in which admissible boundary conditions can be enforced is maximal in a positive cone associated with the differential operator. The equivalence with a formalism based on boundary operators is investigated and practical means to construct these boundary operators are presented. Finally, the link with Friedrichs' formalism and applications to various PDE's are discussed.

1. Introduction

The notion of symmetric positive systems of first-order PDE's has been introduced by Friedrichs in 1958 [4] in an effort to go beyond the traditional classification of PDE's into elliptic, parabolic, and hyperbolic types. Friedrichs wanted to handle PDE's which are partly elliptic and partly hyperbolic using a single functional framework. He introduced a very clever technique to characterize admissible boundary conditions. This technique involves a nonuniquely defined, positive matrix-valued boundary field having peculiar algebraic properties. One difficulty we see in the theory developed by Friedrichs is that it is not intrinsic since the matrix-valued boundary field used to enforce boundary conditions is not uniquely defined. Moreover, the theory involves boundary values of the solution to the PDE whose meaning is not clear. Still, many advances have been reported in the literature to clarify the meaning of traces in Friedrichs' systems; see, among others, [5, 6, 8, 9].

The goal of the present paper is to revisit Friedrichs' theory and to reformulate it so as to avoid invoking traces at the boundary and to remove the arbitrariness referred to above. The theory is expressed in an intrinsic way which does not involve any *ad hoc* matrix-valued boundary field. Furthermore, the theory goes beyond the realm of PDE's since it is formulated in terms of operators acting in abstract Hilbert spaces. As such, it provides sufficient conditions for Hilbert operators to be bijective which are of independent interest.

The paper is organized as follows. In §2 we introduce the notation and we define the problem under consideration. In §3 we present the so-called cone formalism. It consists of a set of geometric conditions that are sufficient to guarantee that the linear Hilbert operator under consideration and its formal adjoint are bijective.

Date: Draft version: September 27, 2005.

1991 *Mathematics Subject Classification.* 47A05, 46C05, 35F05, 35F15.

Key words and phrases. Bijective Hilbert operators – Graph spaces – Maximality – Boundary operators – Partial Differential Equations – Friedrichs' systems .

The main results of this section are the set of conditions (V1)-(V2) together with Theorems 3.1 and 3.3. In §4 we investigate a different formalism that ensures the bijectivity of the linear operator and its formal adjoint. This formalism, which is closely related to the approach followed by Friedrichs in [4] to enforce suitable boundary conditions for systems of first-order PDE's, is based on a boundary operator endowed with *ad hoc* algebraic properties. The main results of this section are the set of conditions (M1)–(M2) together with Theorems 4.2 and 4.3 where the equivalence between the cone formalism and the boundary operator formalism is investigated. Finally, in §5 we show how our formalism relates to that of Friedrichs and we present some applications to systems of first-order PDE's.

2. The setting

2.1. **Definitions and basic properties.** Let L be a Hilbert space equipped with the scalar product $(\cdot,\cdot)_L$ and the corresponding norm $\|\cdot\|_L$. Let $\mathfrak{D}$ be a dense subspace of L. We identify L and its dual; that is, we take L as a pivot space. We assume that we have at hand two linear operators $T : \mathfrak{D} \to L$ and $\tilde{T} : \mathfrak{D} \to L$ such that

$$\text{(T1)} \qquad \forall (\varphi,\psi) \in \mathfrak{D}\times\mathfrak{D}, \quad (T\varphi,\psi)_L = (\varphi,\tilde{T}\psi)_L,$$

$$\text{(T2)} \qquad \exists c, \quad \forall \varphi\in\mathfrak{D}, \quad \|(T+\tilde{T})\varphi\|_L \le c\|\varphi\|_L.$$

In the context of first-order PDE's, property (T2) has nontrivial consequences since it implies the symmetry of the coefficients of the differential operator; see the proof of Proposition 5.1 and property (A2).

Remark 2.1. One may think of $\mathfrak{D}$ as being the set of smooth functions compactly supported in a domain of $\mathbb{R}^d$. One may think of L as being the set of Lebesgue measurable functions whose square is integrable. One may think of T as being a differential operator and of $\tilde{T}$ as being the formal adjoint of T. We do not use the notation T^* to avoid confusion with true adjoints.

Let W_0 be the completion of $\mathfrak{D}$ with respect to the scalar product $(\cdot,\cdot)_L + (T\cdot,T\cdot)_L$. Then, owing to (T1), one can show that, modulo injections, the following identifications can be done:

$$\mathfrak{D} \subset W_0 \subset L \equiv L' \subset W_0' \subset \mathfrak{D}', \tag{2.1}$$

where $\mathfrak{D}'$ is the algebraic dual of $\mathfrak{D}$ and L' and W_0' are topological duals. Observe that owing to (T2), the completion of $\mathfrak{D}$ with respect to the scalar product $(\cdot,\cdot)_L + (\tilde{T}\cdot,\tilde{T}\cdot)_L$ is also W_0. We henceforth abuse the notation by identifying the unique extensions of T and $\tilde{T}$ to W_0 with T and $\tilde{T}$, respectively. Following the terminology of Aubin [1, §5.5], we say that W_0 is the minimal domain of T and $\tilde{T}$. Let us now observe that the true adjoint of $\tilde{T}$, say $(\tilde{T})^* : L \longrightarrow W_0'$, is the unique extension of $T : W_0 \longrightarrow L$. We then again abuse the notation by setting $T = (\tilde{T})^* \in \mathcal{L}(L;W_0')$. Likewise the true adjoint $T^* : L \longrightarrow W_0'$ is the unique extension of $\tilde{T} : W_0 \longrightarrow L$ and we abuse the notation by setting $\tilde{T} = T^* \in \mathcal{L}(L;W_0')$. Since $L \subset W_0'$ it now makes sense to define

$$W = \{v \in L;\, Tv \in L\}. \tag{2.2}$$

Clearly, $W_0 \subset W$.

Lemma 2.1. *Assume* (T1). *Then, W equipped with the so-called graph norm $\|v\|_W = \|v\|_L + \|Tv\|_L$ is a Hilbert space.*

Proof. Let (v_n) be a Cauchy sequence in W. Hence, v_n and Tv_n are Cauchy sequences in L. Let v and w be the corresponding limits in L. Observe that for all $\psi \in W_0$, property (T1) implies $(v_n, \tilde{T}\psi)_L = (Tv_n, \psi)_L$. Hence,

$$\langle Tv, \psi\rangle_{W_0', W_0} = (v, \tilde{T}\psi)_L \leftarrow (v_n, \tilde{T}\psi)_L = (Tv_n, \psi)_L \rightarrow (w, \psi)_L.$$

This shows that $Tv \in L$ and that $Tv = w$. □

W is called the maximal domain of T and is also sometimes referred to as the graph space. The scalar product in W is $(u, v)_W = (u, v)_L + (Tu, Tv)_L$. Note that owing to property (T2) we also have the following characterization:

$$W = \{v \in L;\, \tilde{T}v \in L\}. \tag{2.3}$$

Lemma 2.2. *Assume* (T1)–(T2). *Then, $T + \tilde{T} \in \mathcal{L}(L; L)$ and $T + \tilde{T}$ is self-adjoint on L, i.e.,*

$$\forall (u, v) \in L \times L, \quad ((T + \tilde{T})u, v)_L = ((T + \tilde{T})v, u)_L. \tag{2.4}$$

Proof. That $T+\tilde{T} \in \mathcal{L}(L; L)$ is a consequence of (T2) together with our extending T to $\mathcal{L}(L; W_0')$ and $\tilde{T}$ to $\mathcal{L}(L; W_0')$. To prove the second part of the statement consider $(u, v) \in L \times L$. Since $\mathfrak{D}$ is dense in L, there exist sequences (u_n) and (v_n) in $\mathfrak{D}$ converging to u and v in L, respectively. Owing to (T1), $((T + \tilde{T})u_n, v_n)_L = ((T + \tilde{T})v_n, u_n)_L$. Letting $n \to \infty$ and using the fact that $T+\tilde{T} \in \mathcal{L}(L; L)$ yields (2.4). □

2.2. **The boundary operator D.** Let us now introduce the operator $D \in \mathcal{L}(W; W')$ such that

$$\forall (u, v) \in W \times W, \quad \langle Du, v\rangle_{W', W} = (Tu, v)_L - (u, \tilde{T}v)_L. \tag{2.5}$$

Observe that this definition makes sense since both T and $\tilde{T}$ are in $\mathcal{L}(W; L)$.

Lemma 2.3. *Assume* (T1)–(T2). *Then, D is self-adjoint, i.e.,*

$$\forall (u, v) \in W \times W, \quad \langle Du, v\rangle_{W', W} = \langle Dv, u\rangle_{W', W}. \tag{2.6}$$

Proof. Let $(u, v) \in W \times W$. Then,

$$\begin{aligned}\langle Du, v\rangle_{W', W} - \langle Dv, u\rangle_{W', W} &= (Tu, v)_L - (u, \tilde{T}v)_L - (Tv, u)_L + (v, \tilde{T}u)_L \\ &= ((T + \tilde{T})u, v)_L - ((T + \tilde{T})v, u)_L = 0,\end{aligned}$$

owing to (2.4). □

For all subsets $X \subset W$, we denote by $X^\perp$ the polar set of X, i.e., the set of the continuous linear forms in W' that are zero on X. Similarly, for all subsets $Y \subset W'$, we denote by $Y^\perp$ the polar set of Y, i.e., the set of the continuous linear forms in $W'' \equiv W$ that are zero on Y. A very important result is the following

Lemma 2.4. *Assume* (T1)–(T2). *Then,*

$$\mathrm{Ker}(D) = W_0 \qquad \text{and} \qquad \mathrm{Im}(D) = W_0^\perp. \tag{2.7}$$

In particular, $D(W)$ is closed.

Proof. (1) Let us first prove that $W_0 \subset \mathrm{Ker}(D)$. Let $\psi \in W_0$ and let $v \in W$. Using the fact that the true adjoint of the operator $T : W_0 \longrightarrow L$, say $T^* : L \longrightarrow W_0'$, is the unique extension of $\tilde{T} : W_0 \longrightarrow L$, we infer

$$\langle D\psi, v\rangle_{W',W} = (T\psi, v)_L - (\psi, \tilde{T}v)_L = (T\psi, v)_L - \langle \psi, T^* v\rangle_{W_0,W_0'} = 0,$$

i.e., $\langle D\psi, v\rangle_{W',W} = 0$ for all $v \in W$; as a result, $W_0 \subset \mathrm{Ker}(D)$.
(2) Let us now prove that $W_0^\perp \subset \mathrm{Im}(D)$. Let $x \in W_0^\perp$. Owing to the Riesz representation theorem, there is $z \in W$ such that for all $w \in W$,

$$(z, w)_L + (Tz, Tw)_L = \langle x, w\rangle_{W',W}.$$

For all $v \in W_0$, it is inferred that

$$\langle \tilde{T}Tz, v\rangle_{W_0',W_0} = (Tz, Tv)_L = -(z, v)_L + \langle x, v\rangle_{W',W} = -(z, v)_L,$$

since $x \in W_0^\perp$. In other words, $\tilde{T}Tz$ is in L, meaning that Tz is in W (see (2.3)) and

$$\tilde{T}Tz = -z \quad \text{in } L.$$

Let us set $u = Tz \in W$. Then observing that $\tilde{T}u = -z$, we infer that for all $w \in W$,

$$\begin{aligned}\langle Du, w\rangle_{W',W} = \langle Dw, u\rangle_{W',W} &= (Tw, u)_L - (w, \tilde{T}u)_L \\ &= (Tw, Tz)_L + (w, z)_L = \langle x, w\rangle_{W',W}.\end{aligned}$$

That is to say $Du = x$, i.e., $x \in \mathrm{Im}(D)$.
(3) Owing to Steps 1 and 2, it is inferred that

$$\overline{\mathrm{Im}(D)} \subset W_0^\perp \subset \mathrm{Im}(D),$$

whence (2.7) is easily deduced. □

Remark 2.2. Since $\mathrm{Ker}(D) = W_0$, one may henceforth think of D as a boundary operator. One may also think of the formula

$$(Tu, v)_L = (u, \tilde{T}v)_L + \langle Du, v\rangle_{W',W},$$

as an integration by parts.

3. The cone formalism

Let V and V^* be two subspaces of W. The purpose of this section is to identify sufficient conditions on V and V^* so that the restricted operators $T : V \to L$ and $\tilde{T} : V^* \to L$ are isomorphisms.

3.1. **The two key assumptions.** Let us introduce the following cones:

(3.1) $$C^+ = \{w \in W;\ \langle Dw, w\rangle_{W',W} \geq 0\},$$

(3.2) $$C^- = \{w \in W;\ \langle Dw, w\rangle_{W',W} \leq 0\},$$

(3.3) $$C^0 = C^+ \cap C^- = \{w \in W;\ \langle Dw, w\rangle_{W',W} = 0\}.$$

The two key assumptions on which the cone formalism is based are the following:

(v1) $$V \subset C^+ \text{ and } V^* \subset C^-,$$

(v2) $$V = D(V^*)^\perp \text{ and } V^* = D(V)^\perp.$$

It is straightforward to verify the following

Lemma 3.1. *Assume* (V2). *Then,*

$$V \text{ and } V^* \text{ are closed in } W, \tag{3.4}$$

$$\mathrm{Ker}(D) = W_0 \subset V \cap V^*. \tag{3.5}$$

Remark 3.1. In the context of PDE's, assumption (V1) is usually easy to verify since it relies on algebraic properties of the differential operator and its formal adjoint. The inclusions $V \subset D(V^*)^\perp$ and $V^* \subset D(V)^\perp$ are also straightforward to establish. The converse inclusions are non-trivial since they usually rely on the surjectivity of trace operators; see the examples discussed in §5 for more details.

3.2. **The well–posedness result.** We henceforth solely focus our attention on the class of operators T that are endowed with the following positivity property: There exists $\mu_0 > 0$ such that

$$\forall w \in L, \quad ((T+\tilde{T})w, w)_L \geq 2\mu_0 \|w\|_L^2. \tag{T3}$$

In other words, $T + \tilde{T}$ is L-coercive on L. In the sequel, an operator T satisfying (T1)–(T2)–(T3) is called a *Friedrichs' operator*.

A first important consequence of the above hypotheses is the following

Lemma 3.2. *Assume* (T1)–(T2)–(T3) *and* (V1)–(V2). *Then, T is L-coercive on V and $\tilde{T}$ is L-coercive on V^*.*

Proof. It is clear that (2.5) implies for all $w \in W$,

$$(Tw, w)_L = \tfrac{1}{2}((T+\tilde{T})w, w)_L + \tfrac{1}{2}\langle Dw, w\rangle_{W',W},$$
$$(\tilde{T}w, w)_L = \tfrac{1}{2}((T+\tilde{T})w, w)_L - \tfrac{1}{2}\langle Dw, w\rangle_{W',W}.$$

Hence, owing to property (T3), for all $w \in W$,

$$(Tw, w)_L \geq \mu_0 \|w\|_L^2 + \tfrac{1}{2}\langle Dw, w\rangle_{W',W}, \tag{3.6}$$

$$(\tilde{T}w, w)_L \geq \mu_0 \|w\|_L^2 - \tfrac{1}{2}\langle Dw, w\rangle_{W',W}. \tag{3.7}$$

Conclude using the fact that $V \subset C^+$ and $V^* \subset C^-$. □

The main result of this section is the following

Theorem 3.1. *Assume* (T1)–(T2)–(T3) *and* (V1)–(V2). *Then, the restricted operators $T : V \to L$ and $\tilde{T} : V^* \to L$ are isomorphisms.*

Proof. (1) Let us prove that $T : V \to L$ is an isomorphism. Owing to (3.4), V is a Hilbert space when equipped with the graph norm $\|\cdot\|_W$. Hence, showing that $T : V \to L$ is an isomorphism amounts to proving statement (ii) in Theorem 3.2 below; see [2] for further aspects of this theorem.

(1.a) Proof of (3.8). The L-coercivity of T on V (see Lemma 3.2) implies that $\sup_{v\in L\setminus\{0\}} \frac{(Tu,v)_L}{\|v\|_L} \geq \mu_0 \|u\|_L$. Hence,

$$\left(1 + \frac{1}{\mu_0}\right) \sup_{v \in L\setminus\{0\}} \frac{(Tu, v)_L}{\|v\|_L} \geq \|Tu\|_L + \|u\|_L \geq \|u\|_W.$$

(1.b) Proof of (3.9). Assume that $v \in L$ is such that $(Tu, v)_L = 0$ for all $u \in V$. Then $(Tz, v)_L = \langle z, \tilde{T}v\rangle_{W_0, W_0'} = 0$ for all $z \in W_0$ since $W_0 \subset V$. Hence $\tilde{T}v = 0$ in W_0', i.e., $\tilde{T}v = 0$ in L owing to the density of W_0 in L. Using (2.3), this yields $v \in W$. The definition of D then implies

$$\forall u \in V, \quad \langle Du, v\rangle_{W',W} = 0 - 0 = 0,$$

i.e., $v \in D(V)^\perp$. Hence, $v \in V^*$ owing to (V2). Since $\tilde{T}$ is L-coercive on V^* and $\tilde{T}v = 0$, one readily deduces $v = 0$.
(2) Proceed similarly to prove that $\tilde{T} : V^* \to L$ is an isomorphism. □

Theorem 3.2 (Banach–Nečas–Babuška (BNB))**.** *Let V, L be two Banach spaces, and denote by $\langle \cdot, \cdot \rangle_{L',L}$ the duality pairing between L' and L. The following statements are equivalent:*

(i) *$T \in \mathcal{L}(V; L)$ is bijective.*
(ii) *There exists a constant $\alpha > 0$ such that*

$$\forall u \in V, \quad \sup_{v \in L' \setminus \{0\}} \frac{\langle v, Tu \rangle_{L',L}}{\|v\|_{L'}} \geq \alpha \|u\|_V, \tag{3.8}$$

$$\forall v \in L', \quad (\langle v, Tu \rangle_{L',L} = 0, \ \forall u \in V) \implies (v = 0). \tag{3.9}$$

Corollary 3.1. *Assume* (T1)–(T2)–(T3) *and* (V1)–(V2)*. Let $f \in L$. Then, the following problems are well-posed:*

(i) *Seek $u \in V$ such that $Tu = f$ in L.*
(ii) *Seek $u^* \in V^*$ such that $\tilde{T}u^* = f$ in L.*

Remark 3.2.

(i) The important consequence of Theorem 3.1 is that (V1)–(V2) are sufficient conditions for Friedrichs' operators to be bijective. These two conditions account for admissible boundary conditions. The novelty of our approach with respect to that of Friedrichs (see §5.1) is that we have exhibited an intrinsic characterization of admissible boundary conditions.

(ii) When one is interested in proving the bijectivity of the operator T only, then conditions (V1)–(V2) can be checked by first choosing a space $V \subset C^+$, setting $V^* = D(V)^\perp$, and then verifying that $V^* \subset C^-$ and that $V = D(V^*)^\perp$. In the above presentation V and V^* play symmetric roles to emphasize the fact that the bijectivity of T goes hand in hand with that of $\tilde{T}$, as already pointed out by Friedrichs.

3.3. **The maximality of V.** Let V and V^* be two spaces satisfying (V1)–(V2). We address in this section the issue of the maximality of V in the positive cone C^+ and that of V^* in the negative cone C^-. We prove that whenever two spaces V and V^* satisfy (V1)–(V2), then V and V^* cannot be extended in the positive and negative cones, respectively. This result is related to the proposal made by Lax to enforce "maximal" boundary conditions for dissipative symmetric linear differential operators; see, e.g., Lax and Phillips [7, p. 428] and Friedrichs [4, p. 355].

Before stating the main theorem, we prove some preliminary results.

Lemma 3.3. *Let V and V^* be two spaces satisfying* (V1)–(V2)*. Then,*

$$V \cap C^0 = V^* \cap C^0 = V \cap V^*. \tag{3.10}$$

Proof. (1) Let $v \in V \cap C^0$. Since V is a vector space, for all $u \in V$ and for all $\lambda \in \mathbb{R}$, $\lambda v + u \in V$. Moreover, $V \subset C^+$ implies $\langle D(\lambda v + u), (\lambda v + u) \rangle_{W',W} \geq 0$. Developing and using the fact that $v \in C^0$ and that D is self-adjoint yields

$$2\lambda \langle Du, v \rangle_{W',W} + \langle Du, u \rangle_{W',W} \geq 0.$$

Since $\lambda \in \mathbb{R}$ is arbitrary, we infer that $\langle Du, v \rangle_{W',W} = 0$ for all $u \in V$, i.e., $v \in D(V)^\perp = V^*$ owing to (V2). Hence, $v \in V^* \cap C^0$. As a result, $V \cap C^0 \subset V^* \cap C^0$.

(2) Let $v \in V^* \cap C^0$. Proceeding as before, we infer that $v \in D(V^*)^\perp = V$ owing to (v2). Hence, $V^* \cap C^0 \subset V \cap V^*$.
(3) Let $v \in V \cap V^*$. Then, it is clear that $v \in C^0$. Hence, $V \cap V^* \subset V \cap C^0$. □

Let us now introduce the operator $D_V : V \longrightarrow V'$ such that

$$\forall u, v \in V, \qquad \langle D_V u, v\rangle_{V',V} = \langle Du, v\rangle_{W',W}. \tag{3.11}$$

Clearly D_V is bounded and self-adjoint.

Lemma 3.4. *Let V and V^* be two spaces satisfying* (v1)–(v2)*. Then,*

$$\mathrm{Ker}(D_V) = V \cap V^*. \tag{3.12}$$

Proof. (1) Let v be in $\mathrm{Ker}(D_V) \subset V$. Then $\langle D_V v, w\rangle_{V',V} = \langle v, Dw\rangle_{W,W'} = 0$ for all w in V, that is v is in V^*; hence, $v \in V^* \cap V$. This proves $\mathrm{Ker}(D_V) \subset V^* \cap V$.
(2) Let v be in $V^* \cap V$. Then, $\langle v, Dw\rangle_{W,W'} = \langle D_V v, w\rangle_{V',V} = 0$ for all w in V, meaning that $D_V v = 0$, i.e., v is in $\mathrm{Ker}(D_V)$. Hence, $V^* \cap V \subset \mathrm{Ker}(D_V)$. □

Lemma 3.5. $\mathrm{Im}(D_V)$ *and* $\mathrm{Im}(D_{V^*})$ *are closed.*

Proof. Proceed as in step 2 of the proof of Lemma 2.4. The two key arguments are that V and V^* are closed subspaces of W (see (3.4)), i.e., V and V^* equipped with the scalar product $(\cdot,\cdot)_W$ are Hilbert spaces, and $W_0 \subset V \cap V^*$ (see (3.5)). □

For all $x \in W$, we set $V_x = V + \mathrm{span}(x)$. Similarly, for $y \in W$, we set $V_y^* = V^* + \mathrm{span}(y)$. We are now in a position to state the main result of this section.

Theorem 3.3. *Let V and V^* be two spaces satisfying* (v1)–(v2)*. Then, V is maximal in C^+ (i.e., there is no $x \in W$ such that $V_x \subset C^+$ and V is a proper subspace of V_x) and V^* is maximal in C^- (i.e., there is no $y \in W$ such that $V_y^* \subset C^-$ and V^* is a proper subspace of V_y^*).*

Proof. We only prove the first statement, the proof of the second one being similar. The proof proceeds by contradiction. Assume that there is $x \in W$ such that $V_x \subset C^+$ and V is a proper subspace of V_x, that is $x \notin V$.
(1) For all $v \in V \cap V^* \subset C^0$ and all $\lambda \in \mathbb{R}$, $x + \lambda v \in V_x \subset C^+$ since V_x is a vector space; that is

$$\langle D(x+\lambda v), x+\lambda v\rangle_{W',W} = \langle Dx, x\rangle_{W',W} + 2\lambda\langle Dx, v\rangle_{W',W} \geq 0.$$

This is possible only if $\langle Dx, v\rangle_{W',W} = 0$ for all $v \in V \cap V^*$. Let us define $\phi_x \in V'$ such that $\langle \phi_x, v\rangle_{V',V} = \langle Dx, v\rangle_{W',W}$ for all $v \in V$. Then, owing to Lemma 3.4

$$\langle \phi_x, v\rangle_{V',V} = 0, \quad \forall v \in V \cap V^* = \mathrm{Ker}(D_V),$$

i.e., $\phi_x \in \mathrm{Ker}(D_V)^{\perp_V}$. Here, for a subset $X \subset V$, $X^{\perp_V}$ denotes the set of the continuous linear forms in V' that are zero on X.
(2) Since $\mathrm{Im}(D_V)$ is closed (see Lemma 3.5) and D_V is self-adjoint, Banach's Closed Range Theorem implies $\phi_x \in \mathrm{Ker}(D_V)^{\perp_V} = \mathrm{Im}(D_V^*) = \mathrm{Im}(D_V)$. In other words, there is $v_x \in V$ such that $\phi_x = D_V v_x$. This immediately implies for all $w \in V$,

$$0 = \langle \phi_x - D_V v_x, w\rangle_{V',V} = \langle Dx - Dv_x, w\rangle_{W',W} = \langle x - v_x, Dw\rangle_{W',W}.$$

In other words, $x - v_x \in D(V)^\perp = V^* \subset C^-$. Since $x - v_x \in V_x \subset C^+$, we infer $x - v_x \in C^0$.
(3) Let us set $z = x - v_x$. Clearly $V_z = V_x$ and $z \in C^0$, but $z \notin V$ since $x \notin V$. Since V_z is a vector space, for all $v \in V$ and for all $\lambda \in \mathbb{R}$, $\lambda z + v \in V_z$. Then, owing to the

fact that $z \in C^0$ and D is self-adjoint, we infer that $\langle D(\lambda z+v),(\lambda z+v)\rangle_{W',W} \geq 0$, implying

$$2\lambda\langle Dv,z\rangle_{W',W} + \langle Dv,v\rangle_{W',W} \geq 0.$$

Since $\lambda \in \mathbb{R}$ is arbitrary, we deduce that $\langle Dv,z\rangle_{W',W} = 0$ for all $v \in V$, i.e., $z \in D(V)^\perp = V^*$. Hence, $z \in V^* \cap C_0$, which owing to Lemma 3.3 implies $z \in V \cap V^*$, thereby contradicting the fact that V is a proper subspace of V_z. □

4. The boundary operator M

The purpose of this section is to investigate an alternative way to enforce boundary conditions, namely to define the space V as the kernel of an operator $(D - M)$ where M is a suitably chosen boundary operator. This approach has close links with that proposed originally by Friedrichs to study PDE's; see §5.1 for further details.

4.1. **Properties of the boundary operator M.**

We assume that there exists an operator $M \in \mathcal{L}(W;W')$ such that

(M1) M is positive, i.e., $\langle Mw,w\rangle_{W',W} \geq 0$ for all w in W,

(M2) $W = \mathrm{Ker}(D-M) + \mathrm{Ker}(D+M)$.

Let $M^* \in \mathcal{L}(W;W')$ denote the adjoint operator of M defined as follows: for all $(u,v) \in W \times W$, $\langle M^*u,v\rangle_{W',W} = \langle Mv,u\rangle_{W',W}$.

The main motivation for introducing the above assumptions is the following

Theorem 4.1. *Assume* (M1)–(M2). *Then the restricted operators* $T : \mathrm{Ker}(D-M) \to L$ *and* $\tilde{T} : \mathrm{Ker}(D+M^*) \to L$ *are isomorphisms.*

Proof. This is a corollary of Theorem 3.1 together with Theorem 4.2. □

We now want to investigate the relation existing between the two sets of hypotheses (M1)–(M2) and (V1)–(V2). To this end we derive properties of the boundary operator M that will be used in the next section.

Lemma 4.1. *Assume* (M1)–(M2). *Then,*

$$\mathrm{Ker}(D) = \mathrm{Ker}(M) = \mathrm{Ker}(M^*), \tag{4.1}$$

$$\mathrm{Im}(D) = \mathrm{Im}(M) = \mathrm{Im}(M^*). \tag{4.2}$$

Proof. (1) Let us first verify that (M1) implies $\mathrm{Ker}(M) = \mathrm{Ker}(M^*)$. Let $x \in \mathrm{Ker}(M)$. Then, for all $y \in W$ and all $\lambda \in \mathbb{R}$,

$$\langle M(y+\lambda x),(y+\lambda x)\rangle_{W',W} = \langle My,y\rangle_{W',W} + \lambda\langle My,x\rangle_{W',W} \geq 0.$$

This implies that $\langle My,x\rangle_{W',W} = 0$ for all $y \in W$, i.e., $x \in \mathrm{Ker}(M^*)$. Similarly, one proves $\mathrm{Ker}(M^*) \subset \mathrm{Ker}(M)$.
(2) Let us now verify that (M2) implies $\mathrm{Im}(D) = \mathrm{Im}(M)$. Let $x \in \mathrm{Im}(D)$. Then, there is $y \in W$ such that $x = Dy$. Let $y = y_- + y_+$ with $y_\pm \in \mathrm{Ker}(D \pm M)$. Then, $x = D(y_- + y_+) = M(y_- - y_+)$, showing that $x \in \mathrm{Im}(M)$. Similarly, one proves $\mathrm{Im}(M) \subset \mathrm{Im}(D)$.
(3) Taking the polar sets of $\mathrm{Im}(D)$ and $\mathrm{Im}(M)$ and using the self-adjointness of D, it is inferred that $\mathrm{Ker}(D) = \mathrm{Ker}(M^*)$, and (4.1) then follows from step 1.
(4) Since $\mathrm{Im}(D)$ is closed owing to Lemma 2.4, so is $\mathrm{Im}(M)$; hence, the Closed Range Theorem implies that $\mathrm{Im}(M^*)$ is closed. Finally, taking the polar sets of

Ker(D) and Ker(M), it is inferred that $\mathrm{Im}(D) = \mathrm{Im}(M^*)$, thus completing the proof. ☐

Remark 4.1. Owing to (4.1), $W_0 \subset \mathrm{Ker}(M)$, i.e., M is indeed a boundary operator.

Lemma 4.2. *Assume* (M1)*. Then,* (M2) *holds if, and only if,*

$$W = \mathrm{Ker}(D - M^*) + \mathrm{Ker}(D + M^*). \tag{M3}$$

Proof. Assume (M1)–(M2). Let $w \in W$ and set $w = w_+ + w_-$ with $w_\pm \in \mathrm{Ker}(D \pm M)$. Observing that $D(w_- + w_+) = M(w_- - w_+)$, it is deduced that $D(w_- + w_+) \in \mathrm{Im}(M)$. Owing to (4.2), there is $z \in W$ such that $D(w_- + w_+) = M^*z$. Using Lemma 4.3 below yields $Dz = M^*(w_- + w_+)$. Set $w_\pm^* = \frac{1}{2}(w_- + w_+ \pm z)$. Then, it is easily verified that $w_\pm^* \in \mathrm{Ker}(D \pm M^*)$ and that $w = w_-^* + w_+^*$. Hence, (M3) holds. Proceed similarly to prove the converse statement, namely that (M1) and (M3) imply (M2). ☐

Lemma 4.3. *Assume* (M1)–(M2)*. Then, for all* $(x, y) \in W \times W$*,* $Dx = M^*y$ *if, and only if,* $M^*x = Dy$*.*

Proof. Taking polar sets in (M2) yields

$$\overline{\mathrm{Im}(D - M^*)} \cap \overline{\mathrm{Im}(D + M^*)} = \{0\}. \tag{4.3}$$

Let $(x, y) \in W \times W$ be such that $Dx = M^*y$. Since

$$M^*x - Dy = (D + M^*)(x - y) = (M^* - D)(x + y),$$

it is inferred from (4.3) that $M^*x = Dy$. Proceed similarly to prove the converse statement. ☐

Remark 4.2. Lemma 4.2 shows that the operators M and M^* play symmetric roles.

4.2. **Equivalence with the cone formalism.** The goal of this section is to show that the formalism based on the operator M, namely (M1)–(M2), is somewhat equivalent to the cone formalism introduced in §3, namely (V1)–(V2).

Theorem 4.2. *Assume that* $M \in \mathcal{L}(W; W')$ *satisfies* (M1)–(M2) *and set*

$$V = \mathrm{Ker}(D - M), \tag{4.4}$$

$$V^* = \mathrm{Ker}(D + M^*). \tag{4.5}$$

Then, V *and* V^* *satisfy* (V1)–(V2)*.*

Proof. (1) For all $v \in V$, $\langle Dv, v\rangle_{W',W} = \langle Mv, v\rangle_{W',W} \geq 0$ owing to (M1). Hence, $V \subset C^+$. Similarly, $V^* \subset C^-$. Hence, (V1) holds.
(2) The proof of (V2) is given in [3]. It is restated here for completeness.
(2.a) Let us prove that $D(V)^\perp \subset V^*$. Let $w \in D(V)^\perp$ and let $z \in W$. Owing to (M2), z can be decomposed into $z = z^+ + z^-$ with $z^\pm \in \mathrm{Ker}(D \pm M)$. Then,

$$\begin{aligned}\langle (D + M^*)w, z\rangle_{W',W} &= \langle (D + M^*)w, z^+\rangle_{W',W} + \langle (D + M^*)w, z^-\rangle_{W',W}\\ &= \langle (D + M)z^-, w\rangle_{W',W} = 2\langle Dz^-, w\rangle_{W',W} = 0,\end{aligned}$$

since $z^- \in V$ and $w \in D(V)^\perp$. As a result, $w \in \mathrm{Ker}(D + M^*)$. Hence, $D(V)^\perp \subset V^*$.
(2.b) Conversely, let $w \in V^*$. Let $v \in V$. Using the fact that $Dv = Mv$ yields

$$\langle Dv, w\rangle_{W',W} = \tfrac{1}{2}\langle (D + M)v, w\rangle_{W',W} = \tfrac{1}{2}\langle (D + M^*)w, v\rangle_{W',W} = 0,$$

i.e., $w \in D(V)^\perp$. Hence, $V^* \subset D(V)^\perp$.
(2.c) Proceed similarly to prove $D(V^*)^\perp = V$. ☐

We now investigate the converse of Theorem 4.2.

Theorem 4.3. *Let V and V^* be two spaces satisfying* (V1)–(V2). *Assume that there exist $P \in \mathcal{L}(W;V)$ and $Q \in \mathcal{L}(W;V^*)$ such that*

$$D(v - Pv) = 0, \quad \forall v \in V, \tag{4.6}$$

$$D(v - Qv) = 0, \quad \forall v \in V^*, \tag{4.7}$$

$$DPQ = DQP. \tag{4.8}$$

Define $M \in \mathcal{L}(W;W')$ by setting for all $(u,v) \in W \times W$,

$$\begin{aligned} \langle Mu, v\rangle_{W',W} &= \langle DPu, Pv\rangle_{W',W} - \langle DQu, Qv\rangle_{W',W} \\ &\quad + \langle D(P+Q-PQ)u, v\rangle_{W',W} - \langle Du, (P+Q-PQ)v\rangle_{W',W}. \end{aligned} \tag{4.9}$$

Then, $V = \mathrm{Ker}(D - M)$, $V^* = \mathrm{Ker}(D + M^*)$, and M satisfies (M1)–(M2).

Proof. The proof of Theorem 4.3, which is closely inspired from that presented by Friedrichs [4] in finite dimension, is detailed in the appendix. The key differences are that we need to verify that $D(V)$ is closed and that the existence of the operators P and Q cannot be taken for granted. □

Remark 4.3.

(i) Observe that the first two terms in the right-hand side of (4.9) are the symmetric part of M and that the last two terms are its skew-symmetric part.

(ii) The operator M satisfying (M1)–(M2) and such that (4.4) and (4.5) hold is not necessarily unique. Equation (4.9) provides just one means to construct one operator M explicitly.

Lemma 4.4. *Assume that $V + V^*$ is closed. Then, there exist projectors $P : W \to V$ and $Q : W \to V^*$ such that $PQ = QP$, and $M \in \mathcal{L}(W;W')$ can be constructed by using* (4.9).

Proof. Since W is a Hilbert space and $V + V^*$ is closed, there exists a closed subspace S of W such that $W = (V + V^*) \oplus S$. Furthermore, let V_1 (resp., V_2) be the orthogonal complement of $V \cap V^*$ in V (resp., V^*) with respect to the scalar product associated with the graph norm, i.e., $(\cdot,\cdot)_L + (T\cdot, T\cdot)_L$. Then, it is clear that $V + V^* = V_1 \oplus V_2 \oplus (V \cap V^*)$. As a result,

$$W = V_1 \oplus V_2 \oplus (V \cap V^*) \oplus S. \tag{4.10}$$

For $w \in W$, we denote by $w = w_1 + w_2 + w_3 + w_4$ the decomposition of w induced by (4.10). Then, define for all $w \in W$,

$$Pw = w_1 + w_3 \qquad \text{and} \qquad Qw = w_2 + w_3. \tag{4.11}$$

It is clear that P and Q are projectors onto V and V^*, respectively, and that $PQ = QP$. Hence, hypotheses (4.6)-(4.7)-(4.8) in Theorem 4.3 hold, showing that the projectors P and Q can be employed to construct M according to (4.9). □

It is not yet clear to us whether properties (V1)–(V2) or properties (M1)–(M2) actually imply that $V+V^*$ is closed in W. Two simple instances where this situation occurs are (1) the case where $W = V+V^*$ and (2) the case where $V = V^*$. Although these situations might seem at first glance somewhat simplistic, it turns out that they are relevant to important PDE applications; see §5. These situations are treated in the following

Corollary 4.1. *Assume that* $V+V^*$ *is closed and let* $M \in \mathcal{L}(W;W')$ *be constructed as in Lemma* 4.4.

(i) *If* $W = V + V^*$*, then* M *is self-adjoint, i.e.,* $M = M^*$.
(ii) *If* $V = V^*$*, then* M *is skew-symmetric, i.e.,* $M = -M^*$.

Proof. (1) Assume that $W = V + V^*$. Then, the space S considered in the proof of Lemma 4.4 is trivial, and hence $P + Q - PQ$ is the identity in W. Therefore, the last two terms in (4.9) cancel, yielding

$$\langle Mu, v\rangle_{W',W} = \langle DPu, Pv\rangle_{W',W} - \langle DQu, Qv\rangle_{W',W}, \tag{4.12}$$

i.e., $M = M^*$.
(2) Assume that $V = V^*$. Then, $P = Q$ so that the first two terms in (4.9) cancel. Since $P + Q - QP = P$ in this case, this yields

$$\langle Mu, v\rangle_{W',W} = \langle DPu, v\rangle_{W',W} - \langle DPv, u\rangle_{W',W}, \tag{4.13}$$

i.e., $M = -M^*$. □

Remark 4.4.
(i) In the case where $V = V^*$, the decomposition (4.10) reduces to $W = V \oplus S$ and $P : W \to V$ is the projector such that for all $w \in W$, Pw is the unique solution in V of

$$(T(Pw - w), Tv)_L + (Pw - w, v)_L = 0, \quad \forall v \in V.$$

(ii) We stress the fact that the conclusions of Corollary 4.1 hold only for the particular operator M that is constructed in Lemma 4.4. Since the operator $M \in \mathcal{L}(W;W')$ satisfying (M1)–(M2) is not necessarily unique, it is sometimes possible to construct boundary operators that satisfy (M1)–(M2), but that do not satisfy the conclusions of Corollary 4.1; see Remark 5.3 for an example.

5. Application to PDE's

This section presents some applications of the present theory to PDE's, and in particular to the systems of first-order PDE's considered by Friedrichs in [4]. As particular examples, we consider a scalar hyperbolic PDE (e.g., a transport equation), a scalar elliptic PDE (e.g., the Laplacian), and a system of coupled PDE's associated with the Maxwell equations in the diffusive regime.

5.1. **Friedrichs' formalism.** Let Ω be a bounded, open, and connected Lipschitz domain in $\mathbb{R}^d$. Let $\mathfrak{D}(\Omega)$ be the space of $\mathfrak{C}^\infty$ scalar-valued functions that are compactly supported in Ω. Let m be a positive integer. Set $L = [L^2(\Omega)]^m$ and $\mathfrak{D} = [\mathfrak{D}(\Omega)]^m$. Clearly, $\mathfrak{D}$ is dense in L.

Let $K \in \mathcal{L}(L;L)$ be a bounded linear operator mapping L to L. Let $\{\mathcal{A}^k\}_{1\le k\le d}$ be a family of d locally integrable functions on Ω with values in $\mathbb{R}^{m,m}$. We denote by $\nabla\cdot A : \mathfrak{D} \longrightarrow \mathfrak{D}'$ the operator such that $\nabla\cdot A(v) = (\sum_{k=1}^d \partial_k \mathcal{A}^k)v$ for all $v \in \mathfrak{D}$. We henceforth assume that

(A1) $\quad \forall k \in \{1,\dots,d\},\ \mathcal{A}^k \in [L^\infty(\Omega)]^{m,m} \quad \text{and} \quad \nabla\cdot A \in \mathcal{L}(L;L),$

(A2) $\quad \forall k \in \{1,\dots,d\},\ \mathcal{A}^k = (\mathcal{A}^k)^t$ a.e. in Ω.

(A3) $\quad \exists \mu_0 > 0, \quad K + K^* - \nabla\cdot A \ge \mu_0 I,$

where I is the identity operator in L.

We define an operator T as follows:

$$T : \mathfrak{D} \ni \psi \longmapsto K\psi + \sum_{k=1}^{d} \mathcal{A}^k \partial_k \psi \in L. \tag{5.1}$$

Let $K^* \in \mathcal{L}(L; L)$ be the adjoint operator of K, i.e., for all $(u, v) \in L \times L$, $(Ku, v)_L = (K^* v, u)_L$. Then the formal adjoint of T is given by

$$\tilde{T} : \mathfrak{D} \ni \psi \longmapsto (K^* - (\nabla \cdot A)^*)\psi - \sum_{k=1}^{d} (\mathcal{A}^k)^t \partial_k \psi \in L. \tag{5.2}$$

We purposely did not use the symmetry property (A2) in (5.2) to emphasize the role it plays in the following

Proposition 5.1. *Assume* (A1)–(A2)–(A3). *Let T and $\tilde{T}$ be defined by* (5.1) *and* (5.2)*, respectively. Then,* (T1)–(T2)–(T3) *hold.*

Proof. (1) Property (T1) simply results from an integration by parts.
(2) To prove property (T2) observe that

$$\begin{aligned} \forall \psi \in \mathfrak{D}, \quad (T + \tilde{T})\psi &= (K + K^* - (\nabla \cdot A)^*)\, \psi + \sum_{k=1}^{d} \left(\mathcal{A}^k - (\mathcal{A}^k)^t \right) \partial_k \psi \\ &= (K + K^* - \nabla \cdot A)\, \psi, \end{aligned}$$

owing to the symmetry property (A2). Hence, $T + \tilde{T}$ is bounded on L.
(3) (T3) follows from the above equation and (A3). □

The space W is characterized by

$$W = \{v \in L;\ \textstyle\sum_{k=1}^{d} \mathcal{A}^k \partial_k v \in L\}.$$

Using (A1), one easily verifies that $[H^1(\Omega)]^m$ is a subspace of W.

Let $n = (n_1, \ldots, n_d)^t$ be the unit outward normal to $\partial\Omega$. The usual way of presenting Friedrichs' systems consists of assuming that the fields $\{\mathcal{A}^k\}_{1 \leq k \leq d}$ are smooth enough so that the matrix

$$\mathcal{D} = \sum_{k=1}^{d} n_k \mathcal{A}^k, \tag{5.3}$$

is meaningful at the boundary. Provided $[\mathfrak{C}^1(\overline{\Omega})]^m$ is dense in $[H^1(\Omega)]^m$ and in W, it can be shown that $\mathcal{D}u \in [H^{-\frac{1}{2}}(\partial\Omega)]^m$. Further characterization and regularity results on $\mathcal{D}u$ can be found in [8, 9]; see also [5] and [6]. Also observe that owing to (A2), the following representation of D in terms of $\mathcal{D}$ holds

$$\langle Du, v \rangle_{W', W} = \int_{\partial\Omega} \sum_{k=1}^{d} v^t n_k \mathcal{A}^k u = \int_{\partial\Omega} v^t \mathcal{D} u,$$

whenever u and v and smooth functions.

The key hypothesis introduced by Friedrichs consists of assuming that there exists a matrix-valued field at the boundary, say $\mathcal{M} : \partial\Omega \longrightarrow \mathbb{R}^{m,m}$, such that, a.e. on $\partial\Omega$,

$$\mathcal{M} \text{ is positive, i.e., } (\mathcal{M}\xi, \xi)_{\mathbb{R}^m} \geq 0 \text{ for all } \xi \text{ in } \mathbb{R}^m, \tag{5.4}$$

$$\mathbb{R}^m = \operatorname{Ker}(\mathcal{D} - \mathcal{M}) + \operatorname{Ker}(\mathcal{D} + \mathcal{M}). \tag{5.5}$$

Let $f \in L$. Following the terminology of Friedrichs [3], a solution to the PDE system $Tu = f$ supplemented with the boundary condition $(\mathcal{D} - \mathcal{M})u|_{\partial\Omega} = 0$ is said to be strong if

$$u \in [\mathfrak{C}^1(\overline{\Omega})]^m, \qquad Tu = f, \qquad (\mathcal{D} - \mathcal{M})u|_{\partial\Omega} = 0. \tag{5.6}$$

Likewise, u is said to be a weak solution if

$$u \in L, \qquad (u, \tilde{T}v)_L = (f, v)_L \quad \forall v \in [\mathfrak{C}^1(\overline{\Omega})]^m \text{ s.t. } (\mathcal{D} + \mathcal{M}^t)v|_{\partial\Omega} = 0. \tag{5.7}$$

Friedrichs proved the uniqueness of strong solutions and the existence of weak solutions; see [8] for more details and further results. That with the above formalism it is difficult to obtain existence and uniqueness simultaneously is a consequence of the boundary conditions being expressed explicitly. This observation is one of the reasons that led us to introduce the boundary conditions in the abstract fashion (V1)-(V2).

Remark 5.1. Observe that the definition of T, (5.1), and the hypothesis (A2) are somewhat restrictive. It is possible to consider more general forms for the differential operators T and $\tilde{T}$ within the framework of the present theory. Let $\{\mathcal{A}^k\}_{1\le k\le d}$ and $\{\mathcal{B}^k\}_{1\le k\le d}$ be two families of smooth functions with values in $\mathbb{R}^{m,m}$. Let $K \in \mathcal{L}(L; L)$. Consider the differential operators

$$T : \mathfrak{D} \ni \psi \longmapsto K\psi + \textstyle\sum_{k=1}^d \mathcal{A}^k \partial_k(\mathcal{B}^k\psi) \in L,$$
$$\tilde{T} : \mathfrak{D} \ni \psi \longmapsto K^*\psi - \textstyle\sum_{k=1}^d (\mathcal{B}^k)^t \partial_k((\mathcal{A}^k)^t\psi) \in L,$$

and define the operator

$$\{\!\!\{A, B\}\!\!\} : L \ni v \longmapsto (\textstyle\sum_{k=1}^d \mathcal{A}^k\partial_k\mathcal{B}^k - (\mathcal{B}^k)^t\partial_k(\mathcal{A}^k)^t)v \in L. \tag{5.8}$$

Then,

$$(T + \tilde{T})\psi = (K + K^* - \{\!\!\{A, B\}\!\!\})\psi + \textstyle\sum_{k=1}^d (\mathcal{C}^k - (\mathcal{C}^k)^t)\partial_k\psi,$$

with $\mathcal{C}^k = \mathcal{A}^k\mathcal{B}^k$. Hence, property (T2) holds provided the matrix $\mathcal{C}^k$ is symmetric a.e. in Ω. (T3) holds if there is $\mu_0 > 0$ such that $K + K^* - \{\!\!\{A, B\}\!\!\} \ge \mu_0 I$. Moreover, if there is $\mu_0 > 0$ such that $K + K^* \ge \mu_0 I$, property (T3) holds independently of the smoothness of the two families $\{\mathcal{A}^k\}_{1\le k\le d}$ and $\{\mathcal{B}^k\}_{1\le k\le d}$ when $\{\!\!\{A, B\}\!\!\} = 0$, which is the case, for instance, if either $\mathcal{A}^k = (\mathcal{B}^k)^t$ or $\mathcal{A}^k = -(\mathcal{B}^k)^t$ a.e. in Ω.

5.2. **Scalar hyperbolic PDE's.** Let β be a vector field in $\mathbb{R}^d$ such that $\beta \in [L^\infty(\Omega)]^d$ and $\nabla\cdot\beta \in L^\infty(\Omega)$. Define the inflow boundary $\partial\Omega^-$ and the outflow boundary $\partial\Omega^+$ as follows:

$$\partial\Omega^- = \{x \in \partial\Omega;\, \beta(x)\cdot n(x) < 0\}, \qquad \partial\Omega^+ = \{x \in \partial\Omega;\, \beta(x)\cdot n(x) > 0\}. \tag{5.9}$$

Let μ be a function in $L^\infty(\Omega)$ and assume there exists $\mu_0 > 0$ such that

$$\mu(x) - \tfrac{1}{2}\nabla\cdot\beta(x) \ge \mu_0 > 0 \quad \text{a.e. in } \Omega. \tag{5.10}$$

Consider the advection–reaction equation

$$\mu u + \beta\cdot\nabla u = f, \tag{5.11}$$

with given data $f \in L^2(\Omega)$. This PDE falls into the framework of §5.1 by setting $Kv = \mu v$ for all $v \in L^2(\Omega)$, and $\mathcal{A}^k = \beta^k$ for $k \in \{1, \ldots, d\}$. It is clear that (A1)–(A2)–(A3) hold with $m = 1$. The graph space is

$$W = \{w \in L^2(\Omega);\, \beta\cdot\nabla w \in L^2(\Omega)\}. \tag{5.12}$$

To derive a simple representation of the boundary operator D, we make the following assumptions:

$$\mathfrak{C}^1(\overline{\Omega}) \text{ is dense in } W, \tag{5.13}$$

$$\partial\Omega^- \text{ and } \partial\Omega^+ \text{ are well-separated, i.e., } \mathrm{dist}(\partial\Omega^-, \partial\Omega^+) > 0. \tag{5.14}$$

Note that (5.13) is a regularity hypothesis on Ω; it can be shown to hold if Ω is Lipschitz. Let $L^2(\partial\Omega; |\beta{\cdot}n|)$ be the space of real-valued functions that are square integrable with respect to the measure $|\beta{\cdot}n|\,\mathrm{d}x$ where $\mathrm{d}x$ is the Lebesgue measure on $\partial\Omega$.

Lemma 5.1. *Provided* (5.13)–(5.14) *hold, the trace operator* $\gamma : \mathfrak{C}^1(\overline{\Omega}) \ni v \longrightarrow v \in L^2(\partial\Omega; |\beta{\cdot}n|)$ *extends uniquely to a continuous operator on* W*. Moreover, the operator* D *has the following representation:*

$$\forall (u,v) \in W \times W, \quad \langle Du, v\rangle_{W',W} = \int_{\partial\Omega} uv(\beta{\cdot}n). \tag{5.15}$$

For brevity, the proof, which is given in [3], is not repeated here. It relies on the fact that there exist two non-negative functions ψ^- and ψ^+ in $\mathfrak{C}^1(\overline{\Omega})$ such that

$$\psi^- + \psi^+ = 1 \text{ on } \Omega, \qquad \psi^-|_{\partial\Omega^+} = 0, \qquad \psi^+|_{\partial\Omega^-} = 0. \tag{5.16}$$

Remark 5.2. The hypothesis (5.14), i.e., $\partial\Omega^-$ and $\partial\Omega^+$ are well-separated, is necessary for Lemma 5.1 to hold as shown by the following counterexample. Consider the triangular domain $\Omega = \{(x,y) \in \mathbb{R}^2; |x| < y < 1\}$ and define the vector field $\beta = (y^\alpha, 0)$ where $\alpha > 0$. Clearly $\partial\Omega^- = \{(x,y) \in \mathbb{R}^2; 0 < y = -x < 1\}$ and $\partial\Omega^+ = \{(x,y) \in \mathbb{R}^2; 0 < y = x < 1\}$, i.e., $\mathrm{dist}(\partial\Omega^-, \partial\Omega^+) = 0$. It is proved in [6] that if $\alpha \in (0,1)$, the trace operator γ defined in Lemma 5.1 cannot be extended to a continuous operator on the graph space W with codomain $L^2(\partial\Omega; |\beta{\cdot}n|)$. This means in particular that D cannot have an integral representation, i.e., (5.15) does not hold.

To enforce boundary conditions, set

$$V = \{v \in W; v|_{\partial\Omega^-} = 0\}, \tag{5.17}$$

$$V^* = \{v \in W; v|_{\partial\Omega^+} = 0\}. \tag{5.18}$$

Lemma 5.2. *Let* V *and* V^* *be defined by* (5.17) *and* (5.18)*, respectively. Then,* (V1)–(V2) *hold.*

Proof. The fact that $V \subset C^+$ and $V^* \subset C^-$ directly results from (5.15) and the definition of $\partial\Omega^-$ and $\partial\Omega^+$. Let us now prove that $V = D(V^*)^\perp$. Let $v \in V$ and let $v^* \in V^*$. Then,

$$\langle Dv^*, v\rangle_{W',W} = \int_{\partial\Omega} vv^*(\beta{\cdot}n) = 0,$$

since $v|_{\partial\Omega^-} = 0$, $v^*|_{\partial\Omega^+} = 0$, and $\beta{\cdot}n$ vanishes on the rest of the boundary. Hence, $v \in D(V^*)^\perp$, i.e., $V \subset D(V^*)^\perp$. Conversely, let $v \in D(V^*)^\perp$. Using the fact that $\psi^- v \in V^*$ yields

$$0 = \langle D(\psi^- v), v\rangle_{W',W} = \int_{\partial\Omega} \psi^- v^2(\beta{\cdot}n) = \int_{\partial\Omega^-} v^2(\beta{\cdot}n).$$

As a result, $v|_{\partial\Omega^-} = 0$, i.e., $v \in V$; hence, $D(V^*)^\perp \subset V$. Proceed similarly to establish that $D(V)^\perp = V^*$. □

We now construct the boundary operator M using the techniques presented in §4.2. To illustrate the fact that there exist many possibilities to construct a suitable operator M, we present two techniques, one inspired from Theorem 4.3 and one inspired from Lemma 4.4.

We first apply Theorem 4.3. Consider the partition of unity defined in (5.16) and define the operators

$$P : W \ni w \longmapsto \psi^+ w \in V \qquad \text{and} \qquad Q : W \ni w \longmapsto \psi^- w \in V^*. \tag{5.19}$$

Using the representation (5.15), it is straightforward to check that (4.6)-(4.7)-(4.8) hold. Hence, (4.12) yields the following representation: For all $(u, v) \in W \times W$,

$$\begin{aligned}\langle Mu, v\rangle_{W',W} &= \int_{\partial\Omega} PuPv(\beta{\cdot}n) - \int_{\partial\Omega} QuQv(\beta{\cdot}n)\\ &= \int_{\partial\Omega} (\psi^+)^2 uv(\beta{\cdot}n) - \int_{\partial\Omega} (\psi^-)^2 uv(\beta{\cdot}n)\\ &= \int_{\partial\Omega^+} uv(\beta{\cdot}n) - \int_{\partial\Omega^-} uv(\beta{\cdot}n) = \int_{\partial\Omega} uv|\beta{\cdot}n|.\end{aligned}$$

We now apply Lemma 4.4. Using the partition of unity defined in (5.16), it is clear that for all $w \in W$, $w = (\psi^+ + \psi^-)w = \psi^+ w + \psi^- w$ with $\psi^+ w \in V$ and $\psi^- w \in V^*$. Hence, $W = V + V^*$; that is to say, we can use the projectors defined in (4.11). For all $(u, v) \in W \times W$, let $u = u_1 + u_2 + u_3$ and $v = v_1 + v_2 + v_3$ be the decompositions induced by (4.10) (observe that $S = \{0\}$ since $W = V + V^*$). Then, (4.12) yields the following representation: For all $(u, v) \in W \times W$,

$$\begin{aligned}\langle Mu, v\rangle_{W',W} &= \int_{\partial\Omega} (u_1 + u_3)(v_1 + v_3)(\beta{\cdot}n) - \int_{\partial\Omega} (u_2 + u_3)(v_2 + v_3)(\beta{\cdot}n)\\ &= \int_{\partial\Omega^-} u_1 v_1(\beta{\cdot}n) - \int_{\partial\Omega^+} u_2 v_2(\beta{\cdot}n)\\ &= \int_{\partial\Omega^+} uv(\beta{\cdot}n) - \int_{\partial\Omega^-} uv(\beta{\cdot}n) = \int_{\partial\Omega} uv|\beta{\cdot}n|.\end{aligned}$$

In this example, both approaches yield the same boundary operator M, and M is self-adjoint in accordance with Corollary 4.1.

5.3. **Scalar elliptic PDE's.** Let μ be a positive function in $L^\infty(\Omega)$ uniformly bounded away from zero and consider the PDE

$$-\Delta u + \mu u = f, \tag{5.20}$$

with given data $f \in L^2(\Omega)$. This equation can be written as a system of first-order PDE's by setting

$$\begin{cases} \sigma + \nabla u = 0,\\ \mu u + \nabla{\cdot}\sigma = f. \end{cases} \tag{5.21}$$

This system of PDE's fits into the framework of §5.1 by setting $m = d + 1$, $L = [L^2(\Omega)]^m$, $K(\sigma, u) = (\sigma, \mu u)^t$ for all $(\sigma, u) \in L$ and for all $k \in \{1, \dots, d\}$,

$$\mathcal{A}^k = \left[\begin{array}{c:c} 0 & e^k \\ \hdashline (e^k)^t & 0 \end{array}\right], \tag{5.22}$$

where e^k is the k-th vector in the canonical basis of $\mathbb{R}^d$. It is easily checked that (A1)–(A2)–(A3) hold. The graph space is

$$W = H(\text{div};\Omega) \times H^1(\Omega). \tag{5.23}$$

Since functions in $H^1(\Omega)$ have traces in $H^{\frac{1}{2}}(\partial\Omega)$ and vector fields in $H(\text{div};\Omega)$ have normal traces in $H^{-\frac{1}{2}}(\partial\Omega)$, the boundary operator D has the following representation: For all $((\sigma,u),(\tau,v)) \in W \times W$,

$$\langle D(\sigma,u),(\tau,v)\rangle_{W',W} = \langle \sigma{\cdot}n, v\rangle_{-\frac{1}{2},\frac{1}{2}} + \langle \tau{\cdot}n, u\rangle_{-\frac{1}{2},\frac{1}{2}}, \tag{5.24}$$

where $\langle,\rangle_{-\frac{1}{2},\frac{1}{2}}$ denotes the duality pairing between $H^{-\frac{1}{2}}(\partial\Omega)$ and $H^{\frac{1}{2}}(\partial\Omega)$.

To enforce boundary conditions, one possible choice consists of setting

$$V = V^* = H(\text{div};\Omega) \times H^1_0(\Omega) = \{(\sigma,u) \in W;\, u|_{\partial\Omega} = 0\}. \tag{5.25}$$

This choice corresponds to Dirichlet boundary conditions; Neumann and Robin boundary conditions can be considered as well.

Lemma 5.3. *Let V and V^* be defined by* (5.25). *Then,* (V1)–(V2) *hold.*

Proof. It is readily seen from (5.24) that $V \subset C^0$; hence, (V1) holds. Let us now prove that $V = D(V)^\perp$. Let $(\sigma,u) \in V$. Then, for all $(\tau,v) \in V$,

$$\langle D(\sigma,u),(\tau,v)\rangle_{W',W} = \langle \sigma{\cdot}n, v\rangle_{-\frac{1}{2},\frac{1}{2}} + \langle \tau{\cdot}n, u\rangle_{-\frac{1}{2},\frac{1}{2}} = 0,$$

since $u|_{\partial\Omega} = 0$ and $v|_{\partial\Omega} = 0$. Hence, $V \subset D(V)^\perp$. Conversely, let $(\sigma,u) \in D(V)^\perp$. Let $\theta \in H^{-\frac{1}{2}}(\partial\Omega)$. There exists $\tau_\theta \in H(\text{div};\Omega)$ such that $\tau_\theta{\cdot}n = \theta$ in $H^{-\frac{1}{2}}(\partial\Omega)$. Since $(\tau_\theta,0) \in V$ and $(\sigma,u) \in D(V)^\perp$, it is inferred that

$$0 = \langle D(\tau_\theta,0),(\sigma,u)\rangle_{W',W} = \langle \theta, u\rangle_{-\frac{1}{2},\frac{1}{2}}.$$

Since θ is arbitrary in $H^{-\frac{1}{2}}(\partial\Omega)$, this yields $u|_{\partial\Omega} = 0$, i.e., $u \in V$. □

We now construct the boundary operator M using the technique of Lemma 4.4. Since $V = V^*$, it is inferred from Corollary 4.1 that M is skew-symmetric. Let $\pi : H^1(\Omega) \to H^1_0(\Omega)$ be the projector such that for all $u \in H^1(\Omega)$, πu is the unique solution in $H^1_0(\Omega)$ of

$$(\nabla(\pi u - u), \nabla v)_{L^2(\Omega)} + (\pi u - u, v)_{L^2(\Omega)} = 0, \quad \forall v \in H^1_0(\Omega). \tag{5.26}$$

Let

$$P : W \ni (\sigma,u) \longmapsto (\sigma,\pi u) \in V. \tag{5.27}$$

Then, (4.13) yields

$$\begin{aligned} \langle M(\sigma,u),(\tau,v)\rangle_{W',W} &= \langle D(\sigma,\pi u),(\tau,v)\rangle_{W',W} - \langle D(\tau,\pi v),(\sigma,u)\rangle_{W',W} \\ &= \langle \sigma{\cdot}n, v\rangle_{-\frac{1}{2},\frac{1}{2}} - \langle \tau{\cdot}n, u\rangle_{-\frac{1}{2},\frac{1}{2}}, \end{aligned} \tag{5.28}$$

since $(\pi u)|_{\partial\Omega} = (\pi v)|_{\partial\Omega} = 0$.

Remark 5.3.

(i) Other suitable boundary operators M can be easily designed. For instance,

$$\langle M(\sigma,u),(\tau,v)\rangle_{W',W} = \langle \sigma{\cdot}n, v\rangle_{-\frac{1}{2},\frac{1}{2}} - \langle \tau{\cdot}n, u\rangle_{-\frac{1}{2},\frac{1}{2}} + \alpha \int_{\partial\Omega} uv,$$

where α is an arbitrary nonnegative real, is a suitable choice to enforce Dirichlet boundary conditions. Observe that $M \neq -M^*$ whenever $\alpha \neq 0$.

(ii) Diffusion equations with non-constant tensor-valued diffusion can be handled by using the formalism alluded to in Remark 5.1. Let $\kappa = (\kappa_{kl})_{1\le k,l\le d}$ be a bounded positive definite matrix-valued field defined on Ω whose lowest eigenvalue is uniformly bounded away from zero. Consider the PDE

$$-\nabla\cdot(\kappa^t\kappa\nabla u) + \mu u = f. \tag{5.29}$$

Here, κ is the square root of the diffusion tensor. The natural way to write this PDE in mixed form consists of setting

$$\begin{cases} \sigma + \kappa\nabla u = 0, \\ \mu u + \nabla\cdot(\kappa^t\sigma) = f. \end{cases} \tag{5.30}$$

This system fits the framework described in Remark 5.1 if we set

$$\mathcal{A}^k = \left[\begin{array}{c:c} \kappa & 0 \\ \hdashline 0 & 1 \end{array}\right], \qquad \mathcal{B}^k = \left[\begin{array}{c:c} 0 & e_k \\ \hdashline (\kappa_k)^t & 0 \end{array}\right], \tag{5.31}$$

where $\kappa_1,\dots,\kappa_d$ are the columns of κ. We observe that for all $k \in \{1,\dots,d\}$, $\mathcal{A}^k\mathcal{B}^k = (\mathcal{A}^k\mathcal{B}^k)^t$ and $\{\!\{\mathcal{A}^k,\mathcal{B}^k\}\!\} = 0$, i.e., (T1)–(T2) hold. Note also that without constructing the matrices $\mathcal{A}^k$ and $\mathcal{B}^k$, $k \in \{1,\dots,d\}$, it is clear that (T2) holds by working directly with T and $\tilde{T}$.

5.4. **Maxwell's equations in the diffusive regime.** Let us consider a simplified form of Maxwell's equations in $\mathbb{R}^3$ in the diffusive regime, i.e., when displacement currents are negligible. One interesting difference with the two previous examples is that the boundary operator D does not admit in general a representation involving quantities defined only at the boundary.

Let σ and μ be two positive functions in $L^\infty(\Omega)$ uniformly bounded away from zero, and consider the system of PDE's

$$\begin{cases} \mu H + \nabla\times E = f, \\ \sigma E - \nabla\times H = g, \end{cases} \tag{5.32}$$

with given data $(f,g) \in [L^2(\Omega)]^3 \times [L^2(\Omega)]^3$. This system of PDE's fits into the framework of §5.1 by setting $m = 6$, $L = [L^2(\Omega)]^3\times[L^2(\Omega)]^3$, $K(H,E) = (\mu H, \sigma E)^t$ for all $(H,E) \in L$ and by introducing for $k \in \{1,2,3\}$ the matrices $\mathcal{A}^k \in \mathbb{R}^{6,6}$ given by

$$\mathcal{A}^k = \left[\begin{array}{c:c} 0 & \mathcal{R}^k \\ \hdashline (\mathcal{R}^k)^t & 0 \end{array}\right]. \tag{5.33}$$

The entries of the matrices $\mathcal{R}^k \in \mathbb{R}^{3,3}$ are those of the Levi-Civita permutation tensor, i.e., $\mathcal{R}^k_{ij} = \epsilon_{ikj}$ for $1 \le i,j,k \le 3$. It is easily checked that (A1)–(A2)–(A3) hold. Moreover, the graph space is

$$W = H(\mathrm{curl};\Omega)\times H(\mathrm{curl};\Omega). \tag{5.34}$$

We assume that Ω is smooth enough so that

$$[H^1(\Omega)]^3 \text{ is dense in } H(\mathrm{curl};\Omega). \tag{5.35}$$

Since vector fields in $H(\mathrm{curl};\Omega)$ have tangential traces in $[H^{-\frac12}(\partial\Omega)]^3$, the following formula holds for all $h \in [H^1(\Omega)]^3$ and all $e \in H(\mathrm{curl};\Omega)$,

$$(\nabla\times e, h)_{[L^2(\Omega)]^3} - (e, \nabla\times h)_{[L^2(\Omega)]^3} = \langle (n\times e), h\rangle_{-\frac12,\frac12}.$$

The boundary operator D is defined for all $((H,E),(h,e)) \in W\times W$ as follows:

$$\begin{aligned}\langle D(H,E),(h,e)\rangle_{W',W} &= (\nabla\times E, h)_{[L^2(\Omega)]^3} - (E, \nabla\times h)_{[L^2(\Omega)]^3} \\ &\quad + (H, \nabla\times e)_{[L^2(\Omega)]^3} - (\nabla\times H, e)_{[L^2(\Omega)]^3}.\end{aligned} \tag{5.36}$$

When H and E are smooth, the right-hand side of (5.36) can be interpreted as the boundary integral $\int_{\partial\Omega}(n\times E)\cdot h + (n\times e)\cdot H$.

Let us now define acceptable boundary conditions for (5.32). One possibility (among many others) consists of setting

$$V = V^* = H(\mathrm{curl};\Omega)\times H_0(\mathrm{curl};\Omega). \tag{5.37}$$

Since vector fields in $H(\mathrm{curl};\Omega)$ have tangential traces in $[H^{-\frac{1}{2}}(\partial\Omega)]^3$, couples $(H,E)\in W$ are in V whenever $E\times n|_{\partial\Omega} = 0$.

Lemma 5.4. *For all $e \in H_0(\mathrm{curl};\Omega)$ and all $h \in H(\mathrm{curl};\Omega)$, $(h, \nabla\times e)_{[L^2(\Omega)]^3} - (\nabla\times h, e)_{[L^2(\Omega)]^3} = 0$.*

Proof. Let (h_n) be a sequence in $[H^1(\Omega)]^3$ converging to h in $H(\mathrm{curl};\Omega)$. Then

$$(\nabla\times E, h_n)_{[L^2(\Omega)]^3} - (E, \nabla\times h_n)_{[L^2(\Omega)]^3} = \langle (n\times E), h_n\rangle_{-\frac{1}{2},\frac{1}{2}} = 0.$$

We obtain the desired result by passing to the limit. □

Lemma 5.5. *Let V and V^* be defined by* (5.37)*. Then,* (v1)–(v2) *hold.*

Proof. Let $(H,E)\in V$. Lemma 5.4 implies

$$\langle D(H,E),(H,E)\rangle_{W',W} = 0,$$

i.e., $(H,E)\in C^0$. Hence, $V = V^* \subset C^0$ showing that (v1) holds. Let us now prove that $V = D(V)^\perp$. Using Lemma 5.4, it is clear that $V\subset D(V)^\perp$. Conversely, let $(h,e)\in D(V)^\perp$. Let (H,E) be in $[H^1(\Omega)]^3\times H_0(\mathrm{curl};\Omega)\subset V$. Then

$$0 = \langle D(H,E),(h,e)\rangle_{W',W} = (H,\nabla\times e)_{[L^2(\Omega)]^3} - (\nabla\times H, e)_{[L^2(\Omega)]^3} = \langle (e\times n), H\rangle_{-\frac{1}{2},\frac{1}{2}}.$$

Since H is arbitrary and the traces of vectors fields in $[H^1(\Omega)]^3$ span $[H^{\frac{1}{2}}(\partial\Omega)]^3$, we conclude that $(e\times n)|_{\partial\Omega} = 0$, i.e., $(h,e)\in V$. This proves $D(V)^\perp\subset V$. □

We now construct the boundary operator M using the technique of Lemma 4.4. Since $V = V^*$, it is inferred from Corollary 4.1 that M is skew-symmetric. Let $\hat\pi : H(\mathrm{curl};\Omega) \longrightarrow H_0(\mathrm{curl};\Omega)$ be the projector such that for all $e\in H(\mathrm{curl};\Omega)$, $\hat\pi e$ is the unique solution in $H_0(\mathrm{curl};\Omega)$ of

$$(\nabla\times(\hat\pi e - e), \nabla\times E)_{[L^2(\Omega)]^3} + (\hat\pi e - e, E)_{[L^2(\Omega)]^3} = 0, \quad \forall E\in H_0(\mathrm{curl};\Omega). \tag{5.38}$$

Let

$$P : W \ni (h,e) \longmapsto (h, \hat\pi e)\in V. \tag{5.39}$$

Then, the operator M defined in (4.13) is such that for all $((H,E),(h,e))\in W\times W$,

$$\langle M(H,E),(h,e)\rangle_{W',W} = \langle D(H,\hat\pi E),(h,e)\rangle_{W',W} - \langle D(h,\hat\pi e),(H,E)\rangle_{W',W}. \tag{5.40}$$

Observe that

$$\langle D(H,\hat\pi E),(h,e)\rangle_{W',W} = (H,\nabla\times e)_{[L^2(\Omega)]^3} - (\nabla\times H, e)_{[L^2(\Omega)]^3}, \tag{5.41}$$

since $(\nabla\times(\hat\pi E), h)_{[L^2(\Omega)]^3} - (\hat\pi E, \nabla\times h)_{[L^2(\Omega)]^3} = 0$ owing to Lemma 5.4. Similarly,

$$\langle D(h,\hat\pi e),(H,E)\rangle_{W',W} = (h,\nabla\times E)_{[L^2(\Omega)]^3} - (\nabla\times h, E)_{[L^2(\Omega)]^3}. \tag{5.42}$$

As a result, it is finally inferred that

$$\langle M(H,E),(h,e)\rangle_{W',W} = -(\nabla\times E,h)_{[L^2(\Omega)]^3} + (E,\nabla\times h)_{[L^2(\Omega)]^3} + (H,\nabla\times e)_{[L^2(\Omega)]^3} - (\nabla\times H,e)_{[L^2(\Omega)]^3}. \tag{5.43}$$

6. Appendix: proof of Theorem 4.3

(1) We first prove that V and V^* are characterized by the following:

$$(\langle Dv,Qw\rangle_{W',W}=0,\ \forall w\in V^*) \Longleftrightarrow (v\in V), \tag{6.1}$$

$$(\langle Dv,Pw\rangle_{W',W}=0,\ \forall w\in V) \Longleftrightarrow (v\in V^*). \tag{6.2}$$

Indeed, let $v\in W$ be such that $\langle Dv,Qw\rangle_{W',W}=0$, $\forall w\in V^*$. Owing to (4.7) and the self-adjointness of D, this implies

$$\langle v,Dw\rangle_{W',W} = \langle v,D(w-Qw)\rangle_{W',W} + \langle Dv,Qw\rangle_{W',W} = 0.$$

Hence, $v\in D(V^*)^\perp = V$ owing to (v2). The converse is evident. The proof of (6.2) is similar.
(2) Since $Pv\in V$ and $Qw\in V^* = D(V)^\perp$ for all $(v,w)\in W\times W$, the following identities hold:

$$\langle DPv,Qw\rangle_{W',W}=0, \quad \forall(v,w)\in W\times W, \tag{6.3}$$

$$D(I-P)P=0, \tag{6.4}$$

$$D(I-Q)Q=0. \tag{6.5}$$

(3) Let us now verify that $\mathrm{Ker}(D)\subset\mathrm{Ker}(M)$. Let $u\in\mathrm{Ker}(D)$, then $u\in V\cap V^*$ owing to (3.5). Using (4.6) and (4.7) yields $DPu=DQu=Du=0$ in W'. In addition, since Qu is in $\mathrm{Ker}(D)\subset V$, (4.6) implies $DPQu=DQu=0$. As a result, all the terms in the right-hand side of (4.9) vanish, i.e., $u\in\mathrm{Ker}(M)$.
(4) Let us prove $V=\mathrm{Ker}(D-M)$.
(4.a) Let $v\in V$ and let us show that $(D-M)v=0$. Observe first that (4.6) implies $v-Pv\in\mathrm{Ker}(D)$. Hence, $v-Pv\in\mathrm{Ker}(M)$ owing to Step 1. This yields $(D-M)v=(D-M)Pv$. Let us prove that $(D-M)Pv=0$ in W'. The definition of M implies that for all $w\in W$, the following holds:

$$\begin{aligned}\langle (D-M)Pv,w\rangle_{W',W} &= \langle DPv,w\rangle_{W',W} - \langle DPPv,Pw\rangle_{W',W} + \langle DQPv,Qw\rangle_{W',W} \\ &\quad - \langle D(P+Q-PQ)Pv,w\rangle_{W',W} + \langle DPv,(P+Q-PQ)w\rangle_{W',W}.\end{aligned}$$

The second term in the right-hand side is equal to $\langle DPv,Pw\rangle_{W',W}$ owing to (6.4); the third vanishes owing to (4.8) and (6.3); for the fourth term observe that $\langle D(Q-PQ)Pv,w\rangle_{W',W} = \langle DQ(I-P)Pv,w\rangle_{W',W}$ and since $v'=(I-P)Pv\in\mathrm{Ker}(D)\subset V^*$ owing to (6.4), it comes that $DQv'=Dv'$ owing to (4.7), then $Dv'=D(I-P)Pv=0$ owing to (6.4) again; finally, the fifth term is equal to $\langle DPv,Pw\rangle_{W',W}$ since $\langle DPv,(Q-PQ)w\rangle_{W',W} = \langle Pv,DQ(I-P)w\rangle_{W',W}=0$ owing to (4.8) and (6.3). Hence,

$$\begin{aligned}\langle (D-M)Pv,w\rangle_{W',W} &= \langle DPv,w\rangle_{W',W} - \langle DPv,Pw\rangle_{W',W} \\ &\quad - \langle DPv,w\rangle_{W',W} + \langle DPv,Pw\rangle_{W',W} = 0.\end{aligned}$$

Therefore, $(D-M)v=0$ in W', i.e., $v\in\mathrm{Ker}(D-M)$. This shows that $V\subset\mathrm{Ker}(D-M)$.

(4.b) Let us show the converse. Let $v \in \mathrm{Ker}(D-M)$ and let us prove that $v \in V$. Let $w \in V^*$. A direct calculation yields

$$\langle (D-M)v, Qw\rangle_{W',W} = \langle Dv, Qw\rangle_{W',W} - \langle DPv, PQw\rangle_{W',W} + \langle DQv, QQw\rangle_{W',W} - \langle D(P+Q-PQ)v, Qw\rangle_{W',W} + \langle Dv, (P+Q-PQ)Qw\rangle_{W',W}.$$

By using arguments similar to those that have been used in step 4.a, we infer that the second term in the right-hand side vanishes; the third is equal to $\langle DQv, Qw\rangle_{W',W}$; the fourth reduces to $\langle DQv, Qw\rangle_{W',W}$; and the fifth is equal to $\langle Dv, Qw\rangle_{W',W}$. Hence,

$$\langle (D-M)v, Qw\rangle_{W',W} = 2\langle Dv, Qw\rangle_{W',W}.$$

Since $v \in \mathrm{Ker}(D-M)$, the above equation yields that $\langle Dv, Qw\rangle_{W',W} = 0$, $\forall w \in V^*$. Hence, the characterization (6.1) yields $v \in V$. Therefore, $\mathrm{Ker}(D-M) \subset V$.

(5) The proof of $V^* = \mathrm{Ker}(D+M^*)$ is similar to that of $V = \mathrm{Ker}(D-M)$. We do not repeat the arguments.

(6) Proof of (M1). For all $w \in W$, (4.9) yields

$$\langle Mw, w\rangle_{W',W} = \langle DPw, Pw\rangle_{W',W} - \langle DQw, Qw\rangle_{W',W} \geq 0,$$

since $V \subset C^+$ and $V^* \subset C^-$ owing to (V1).

(7) Proof of (M2). Observe first that $(D+M)(W) \subset (V^*)^\perp$. Indeed, for all $(w,v) \in W \times V^*$,

$$\langle (D+M)w, v\rangle_{W',W} = \langle (D+M^*)v, w\rangle_{W',W} = 0,$$

since $V^* = \mathrm{Ker}(D+M^*)$. Furthermore, $D(V)$ is closed (see step 8 below); hence, $D(V) = (V^*)^\perp$ owing to (V2). As a result, $(D+M)(W) \subset D(V)$. Let $w \in W$. Owing to the above inclusion, there exists $v_w \in V$ such that $Dv_w = \frac{1}{2}(D+M)w$. To conclude, observe that $w = v_w + (w - v_w)$, $v_w \in \mathrm{Ker}(D-M)$, and $w - v_w \in \mathrm{Ker}(D+M)$ since

$$(D+M)(w-v_w) = 2Dv_w - (D+M)v_w = 2Dv_w - 2Dv_w = 0.$$

(8) To complete the proof, we need to prove that $D(V)$ is closed. Let (v_n) be a sequence in V such that Dv_n is Cauchy in W'. Since $D(W)$ is closed (see Lemma 2.4) there is v in W such that $Dv_n \to Dv$ in W'. Let $w \in V^*$, then

$$\langle Dw, v\rangle_{W',W} = \langle Dv, w\rangle_{W',W} = \lim_{n\to+\infty} \langle Dv_n, w\rangle_{W',W} = \lim_{n\to+\infty} \langle v_n, Dw\rangle_{W',W} = 0.$$

Hence v is in $D(V^*)^\perp = V$.

Acknowledgment. The authors are thankful to E. Cancès (CERMICS, ENPC) for fruitful discussions.

References

[1] J.-P. Aubin. *Applied Functional Analysis*. Pure and Applied Mathematics. Wiley-Interscience, New York, second edition, 2000.

[2] A. Ern and J.-L. Guermond. *Theory and Practice of Finite Elements*, volume 159 of *Applied Mathematical Sciences*. Springer-Verlag, New York, NY, 2004.

[3] A. Ern and J.-L. Guermond. Discontinuous Galerkin methods for Friedrichs' symmetric systems. I. General theory. *SIAM J. Numer. Anal.*, 2005. submitted (internal report available at `cermics.enpc.fr/reports/CERMICS-2005/CERMICS-2005-276.pdf`).

[4] K.O. Friedrichs. Symmetric positive linear differential equations. *Comm. Pure Appl. Math.*, 11:333–418, 1958.

[5] P. Houston, J. A. Mackenzie, E. Süli, and G. Warnecke. A posteriori error analysis for numerical approximations of Friedrichs systems. *Numer. Math.*, 82(3):433–470, 1999.

[6] M. Jensen. *Discontinuous Galerkin Methods for Friedrichs Systems with Irregular Solutions*. PhD thesis, University of Oxford, 2004.

[7] P.D. Lax and R.S. Phillips. Local boundary conditions for dissipative symmetric linear differential operators. *Comm. Pure Appl. Math.*, 13:427–455, 1960.

[8] J. Rauch. Symmetric positive systems with boundary characteristic of constant multiplicity. *Trans. Amer. Math. Soc.*, 291(1):167–187, 1985.

[9] J. Rauch. Boundary value problems with nonuniformly characteristic boundary. *J. Math. Pures Appl. (9)*, 73(4):347–353, 1994.

[1]CERMICS, Ecole nationale des ponts et chaussées, Champs sur Marne, 77455 Marne la Vallée Cedex 2, France

[2]LIMSI (CNRS-UPR 3152), BP 133, 91403, Orsay, France. Current address: Dept. Math, Texas A&M, College Station, TX 77843-3368, USA

E-mail address: ern@cermics.enpc.fr

E-mail address: guermond@math.tamu.edu

E-mail address: caplain@cermics.enpc.fr