M661: Mathematical Theory of Finite Elements. Final TEST, Dec 4th, 2007 (To be returned Dec 11th).

Let Ω be a bounded polygon in $\mathbb{R}^2$. Let f be a function in $L^2(\Omega)$ and β be a constant vector field. Let $\partial\Omega^-$ be the inflow boundary. Consider the problem

$$u + \beta\cdot\nabla u = f, \quad u|_{\partial\Omega^-} = 0$$

Question 1: Give a weak formulation of the problem using $W_0 = \{v \in L^2(\Omega),\ \beta\cdot\nabla v \in L^2(\Omega),\ v|_{\partial\Omega^-} = 0\}$ as solution space and $L^2(\Omega)$ as test space.

Question 2: Give the so-called Least-Squares (LS) formulation of the problem using W_0 as both solution space and test space.

Question 3: What is the PDE the LS formulation solves? What are the boundary conditions?

Question 4: Prove well-posedness of the Least-Squares formulation.

Question 5: Let $\{\mathcal{T}_h\}_{h>0}$ be a regular conforming mesh family composed of triangles. Denote $W_h = \{v_h \in \mathcal{C}^0(\overline{\Omega}),\ v_h|_K \in \mathbb{P}_k,\ \forall K \in \mathcal{T}_h,\ v_h|_{\partial\Omega^-} = 0\}$, where $k \geq 1$. Write the Least-Squares formulation of the problem in W_h.

Question 6: Assuming that Ω is the segment $(0, 1)$, the mesh is uniform, and $k = 1$ (only in this question), derive the linear system that is satisfied by the nodal values of the approximate solution of the LS formulation.

Question 7: Derive a error estimate on the L^2-norm and the graph-norm assuming that the solution is in $H^{k+1}(\Omega) \cap W_0$.

Question 8: Give the Galerkin Least-Squares (GaLS) formulation of the problem.

Question 9: Assuming that Ω is the segment $(0, 1)$, the mesh is uniform, and $k = 1$ (only in this question), derive the linear system that is satisfied by the nodal values of the approximate solution of the GaLS formulation.

Question 10: Derive a error estimate on the L^2-norm and the graph-norm assuming that the solution is in $H^{k+1}(\Omega) \cap W_0$.

Question 11: Did you find a difference on the a priori error estimates between the two methods? Explain which of the two methods should be preferred according to you.